OXFORD IB DIPLOMA PROGRAMME

WORKED SOLUTIONS

MATHEMATICS HIGHER LEVEL

COURSE COMPANION

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1 Mathematics as the science of patterns

Answers

Skills check

1 **a** $\{1, 2, 3, 4, 5\}$ **b** $\{-4, -3, -2, -1, 0, 1\}$

c $\{1, 2, 3, 4, 5, 6\}$

2 **a**
$$\begin{aligned} 3(x-4)-2(x+7)&=0\\ 3x-12-2x-14&=0\\ x&=26 \end{aligned}$$

b
$$\begin{aligned} 3x-2(2x+5)&=2\\ 3x-4x-10&=2\\ -x&=12\\ x&=-12 \end{aligned}$$

c
$$\begin{aligned} 5x+4-2(x+6)&=x-(3x-2)\\ 5x+4-2x-12&=x-3x+2\\ 3x-8&=-2x+2\\ 5x&=10\\ x&=2 \end{aligned}$$

3 **a**
$$\begin{aligned} 2(\sqrt{3}-2)+\sqrt{3}(1-\sqrt{3})&=2\sqrt{3}-4+\sqrt{3}-3\\ &=3\sqrt{3}-7 \end{aligned}$$

b $\frac{3}{\sqrt{2}}+5\sqrt{2}=\frac{3\sqrt{2}}{2}+5\sqrt{2}=\frac{13}{2}\sqrt{2}$

c
$$\begin{aligned} \frac{(1+\sqrt{3})}{(1-\sqrt{3})}&=\frac{(1+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}=\frac{1+\sqrt{3}+\sqrt{3}+3}{1-3}\\ &=\frac{4+2\sqrt{3}}{-2}=-2-\sqrt{3} \end{aligned}$$

4 **a**
$$\begin{aligned} \frac{1}{(x-2)}&=\frac{-3}{(1-2x)}\\ 1-2x&=-3(x-2)\\ 1-2x&=-3x+6\\ x&=5 \end{aligned}$$

b
$$\begin{aligned} \frac{2x}{2x^2+1}&=\frac{1}{x-1}\\ 2x(x-1)&=2x^2+1\\ 2x^2-2x&=2x^2+1\\ -2x&=1\\ x&=-\tfrac{1}{2} \end{aligned}$$

5 **a** 35 **b** -10

Exercise 1A

1 **a** $0, 1.5, 3$

b $\frac{9}{10}, \frac{11}{12}, \frac{13}{14}$

c $\frac{1}{99}, \frac{1}{143}, \frac{1}{195}$

(denominators can be written as 1×3, 3×5, 5×7, 7×9, 9×11, 11×13, 13×15)

2 **a** $r(r+1)$

b $\frac{1}{r^2+1}$

c $2r-3$

3 **a** $1, 5, 9, 13$

b $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$

c $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$

4 **a** $2+6+12+20$

b $\frac{1}{3}+\frac{2}{5}+\frac{3}{7}+\frac{4}{9}+\frac{5}{11}$

c $-1+4-9+16-25$

5 **a** $\sum_{r=1}^{\infty} 4r-5$

b $\sum_{r=1}^{10} (-1)^r$

c $\sum_{r=1}^{6} 6(-2)^{r-1}$

Investigation – quadratic sequences

$N = n^2 - 2n + 3$

$$\begin{aligned} n=p-1 \Rightarrow n^2-2n+3&=(p-1)^2-2(p-1)+3\\ &=p^2-2p+1-2p+2+3\\ &=p^2-4p+6\\ n=p \qquad \Rightarrow n^2-2n+3&=p^2-2p+3\\ n=p+1 \Rightarrow n^2-2n+3&=(p+1)^2-2(p+1)+3\\ &=p^2+2p+1-2p-2+3\\ &=p^2+2 \end{aligned}$$

first differences are $2p-3$ and $2p-1$

second difference $=(2p-1)-(2p-3)=2$ (a constant)

$N = 2n^2 + 2n + 1$

$$\begin{aligned} n=p-1 \Rightarrow 2n^2+2n+1&=2(p-1)^2+2(p-1)+1\\ &=2p^2-4p+2+2p-2+1\\ &=2p^2-2p+1\\ n=p \qquad \Rightarrow 2n^2+2n+1&=2p^2+2p+1\\ n=p+1 \Rightarrow 2n^2+2n+1&=2(p+1)^2+2(p+1)+1\\ &=2p^2+4p+2+2p+2+1\\ &=2p^2+6p+5 \end{aligned}$$

first differences are $4p$ and $4p+4$

second difference $=4$ (a constant)

$N = -n^2 + 3n - 4$

$$\begin{aligned}
n = p - 1 &\Rightarrow -n^2 + 3n - 4 = -(p-1)^2 + 3(p-1) - 4\\
&\qquad\qquad\qquad = -p^2 + 2p - 1 + 3p - 3 - 4\\
&\qquad\qquad\qquad = -p^2 + 5p - 8\\
n = p &\Rightarrow -n^2 + 3n - 4 = -p^2 + 3p - 4\\
n = p + 1 &\Rightarrow -n^2 + 3n - 4 = -(p+1)^2 + 3(p+1) - 4\\
&\qquad\qquad\qquad = -p^2 - 2p - 1 + 3p + 3 - 4\\
&\qquad\qquad\qquad = -p^2 + p - 2
\end{aligned}$$

first differences are $-2p + 4$ and $-2p + 2$

second difference $= (-2p + 2) - (-2p + 4)$

$= -2$ (a constant)

Conjecture: For the quadratic $N = an^2 + bn + c$ the second difference is a constant and is equal to $2a$.

Proof:

$$\begin{aligned}
n = p - 1 &\Rightarrow an^2 + 6n + c = a(p-1)^2 + b(p-1) + c\\
&\qquad\qquad\qquad = ap^2 - 2ap + a + bp - b + c\\
n = p &\Rightarrow an^2 + bn + c = ap^2 + bp + c\\
n = p + 1 &\Rightarrow an^2 + bn + c = a(p+1)^2 + b(p+1) + c\\
&\qquad\qquad\qquad = ap^2 + 2ap + a + bp + b + c
\end{aligned}$$

first differences are $2ap - a + b$ and $2ap + a + b$

second difference $= 2a$, which proves the conjecture.

Investigation – triangular numbers

Since the second difference is a constant (1) the triangle numbers can be generated by a quadratic

$$\begin{aligned}
&N = an^2 + bn + c \quad 2a = 1 \Rightarrow a = \tfrac{1}{2}\\
&\qquad N = \tfrac{1}{2}n^2 + bn + c\\
&n = 1 \Rightarrow \tfrac{1}{2} + b + c = 1 \Rightarrow b + c = \tfrac{1}{2}\\
&n = 2 \Rightarrow 2 + 2b + c = 3 \Rightarrow 2b + c = 1\\
&\qquad \therefore b = \tfrac{1}{2},\ c = 0\\
&\quad N = \tfrac{1}{2}n^2 + \tfrac{1}{2}n \quad \text{or} \quad N = \tfrac{1}{2}n(n+1)
\end{aligned}$$

Investigation – more number patterns

Square numbers: $N = n^2$

Pentagonal numbers: $N = \frac{n(3n-1)}{2}$

Hexagonal numbers: $N = n(2n - 1)$

Heptagonal numbers: $N = \frac{n(5n-3)}{2}$

Polygonal numbers	N
triangle	$\frac{1}{2}n\,(n+1) = \frac{n}{2}(n+1)$
square	$n^2 = \frac{n}{2}(2n + 0)$
pentagon	$\frac{n(3n-1)}{2} = \frac{n}{2}(3n - 1)$
hexagon	$n(2n - 1) = \frac{n}{2}(4n - 2)$
heptagon	$\frac{n(5n-3)}{2} = \frac{n}{2}(5n - 3)$

Conjecture: For a polygon with k sides the polygonal numbers are given by

$N = \frac{n}{2}\,[(k - 2)\,n - (k - 4)]$

Exercise 1B

1 **a** $u_n = 5 + (n-1)6$

$u_n = 6n - 1$

b $u_n = 10 + (n-1)(-7)$

$u_n = -7n + 17$

c $u_n = a + (n-1)2$

$u_n = 2n + a - 2$

2 **a** $u_{15} = 2 + 14d = 2 + 14 \times 9 = 128$

b $u_{12} = -1 + 11d = -1 + 11 \times \frac{5}{4} = \frac{51}{4}$

c $u_n = 3 + (n-1)4 = 4n - 1$

3 $a + 3d = 18 \Rightarrow a - 15 = 18 \Rightarrow a = 33$

$u_n = 33 + (n-1)(-5) = 38 - 5n$

4 $a + 3d = 0 \qquad (1)$

$a + 13d = 40 \qquad (2)$

$(2) - (1) \Rightarrow 10d = 40 \Rightarrow d = 4$

$\therefore\ a + 12 = 0$ and $a = -12$

5 Salary after 15 years $= u_{16} = a + 15d$

$= 48\,000 + 15 \times 500$

$= €\ 55\,500$

Need $n \times 500 = 24000$

$\Rightarrow n = 48$ years

Exercise 1C

1 **a** $u_1 = 6 \quad d = 13 \quad u_n = 110$

$$\begin{aligned}
6 + (n-1)13 &= 110\\
(n-1)13 &= 104\\
n - 1 &= 8\\
n &= 9
\end{aligned}$$

$S_9 = \frac{9}{2}(6 + 110) = 522$

b $u_1 = 52 \quad d = -11 \quad u_n = -25$

$$\begin{aligned}
52 + (n-1)(-11) &= -25\\
(n-1)(-11) &= -77\\
n - 1 &= 7\\
n &= 8
\end{aligned}$$

$S_8 = \frac{8}{2}(52 - 25) = 108$

c $u_1 = -78 \quad d = -4 \quad u_n = -142$

$$\begin{aligned}
-78 + (n-1)(-4) &= -142\\
(n-1)(-4) &= -64\\
n - 1 &= 16\\
n &= 17
\end{aligned}$$

$S_{17} = \frac{17}{2}(-78 - 142) = -1870$

2 **a** $\sum_{r=1}^{10} 5r + 7 = 12 + 17 + 22 + \ldots\ldots + 57$

$= \frac{10}{2}(12 + 57)$

$= 345$

b $\sum_{r=1}^{15} 5-3r = 2 - 1 - 4 \ldots\ldots - 40$

$= \frac{15}{2}(2 - 40)$

$= -285$

3 $u_1 = 60 \quad u_{10} = -3 \quad n = 16$

$60 + 9d = -3$

$9d = -63$

$d = -7$

$S_{16} = \frac{16}{2}(2 \times 60 + 15 \times -7) = 120$

4 $S_5 = 25 \quad u_4 = 8$

Let the numbers be

$u - 2d, u - d, u, u + d, u + 2d$

$S_5 = u - 2d + u - d + u + u + d + u + 2d$

$\therefore\ 5u = 25$

$u = 5$

$u_4 = 8 \Rightarrow u + d = 8 \Rightarrow d = 3$

The numbers are −1, 2, 5, 8,11

5 $S_n = n(2n + 3)$

$S_1 = 1(2 + 3) = 5 \quad \therefore\ u_1 = 5$

$S_2 = 2(4 + 3) = 14 \quad \therefore\ u_1 + u_2 = 14 \ \therefore\ u_2 = 9$

$\therefore\ d = 4$

$u_1 = 5, \quad u_2 = 9, \quad u_3 = 13, \quad u_4 = 17$

Exercise 1D

1 **a** $u_1 = 1 \ \ r = 2 \ \ u_6 = 2^5 = 32 \quad u_n = 2^{n-1}$

b $u_1 = 9 \ \ r = \frac{1}{3} \ \ u_6 = 9\left(\frac{1}{3}\right)^5 = \frac{1}{27} \ \ u_n = 9\left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^{n-3}$

c $u_1 = x^3 \ r = \frac{1}{x} \ \ u_6 = x^3\left(\frac{1}{x}\right)^5 = \frac{1}{x^2} \ \ u_n = x^3\left(\frac{1}{x}\right)^{n-1} = \left(\frac{1}{x}\right)^{n-4}$

2 **a** $r = \frac{1}{2}, u_{10} = a\,r^9 = 48 \times \frac{1}{512} = \frac{3}{32}$

b $r = -\frac{8}{9} \div \frac{16}{3} = \frac{8}{9} \times \frac{3}{16} = -\frac{1}{6},$

$u_5 = a\,r^4 = \frac{16}{3} \times \frac{1}{1296} = \frac{1}{3 \times 81} = \frac{1}{243}$

3 **a** $a = 0.03, r = 2$

$\Rightarrow 0.03 \times 2^{n-1} = 1.92 \Rightarrow 2^{n-1} = 64 \Rightarrow n = 7$

b $a = 81, r = \frac{1}{3}$

$81 \times \left(\frac{1}{3}\right)^{n-1} = \frac{1}{81} \Rightarrow \left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^8 \Rightarrow n = 9$

4 $a\,r^2 = 2 \qquad (1)$

$a\,r^4 = 18 \qquad (2)$

$(2) \div (1) \Rightarrow r^2 = 9 \Rightarrow r = \pm 3$

$u_2 = a\,r = \frac{2}{9} \times \pm 3 = \pm\frac{2}{3}$

5 $16r^4 = 9 \Rightarrow r^4 = \frac{9}{16} \Rightarrow r = \pm\frac{\sqrt{3}}{2}$

$\Rightarrow u_7 = 16r^6 = 16 \times \frac{27}{64} = \frac{27}{4}$

6 $r = \frac{a+2}{a-4} = \frac{3a+1}{a+2} \Rightarrow a^2 + 4a + 4 = 3a^2 - 11a - 4$

$\Rightarrow 0 = 2a^2 - 15a - 8$

$= (2a + 1)(a - 8)$

$\Rightarrow a = -\frac{1}{2}$ or 8

Hence $r = \frac{1\frac{1}{2}}{-4\frac{1}{2}} = -\frac{1}{3}$ or $r = \frac{10}{4} = \frac{5}{2}$

Exercise 1E

1 **a** $S_6 = \frac{2\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}} = 3.9375$ or $\frac{63}{16}$

b $S_8 = \frac{2\left(1-(-1.5)^8\right)}{1-(-1.5)} = -19.7$ (3 sf) or $\frac{-1261}{64}$

c Sum $= 1 + \frac{\frac{1}{2}\left(1-\left(\frac{-1}{2}\right)^9\right)}{1-\left(\frac{-1}{2}\right)} = 1.33$ (3 sf) or $\frac{683}{512}$

d $u_1 = 0.1, r = 0.2$

Sum $= \frac{0.1\left(1-0.2^{15}\right)}{1-0.2} = \frac{1}{8}(1 - 0.2^{15})$

$= \frac{1}{8}(1 - \frac{1}{5^{15}})$

$= 0.125$ (3 sf)

2 **a** $\sum_{r=0}^{5} 5^{3-r} = 5^3 + 5^2 + 5^1 + 5^0 + 5^{-1} + 5^{-2}$

$= \frac{125\left(1-\left(\frac{1}{5}\right)^6\right)}{1-\frac{1}{5}}$

$= 156.24$ or $\frac{3906}{25}$

b $\sum_{r=0}^{n-1} 9 \times 10^r = 9 + 9 \times 10 + 9 \times 10^2 + \ldots + 9 \times 10^{n-1}$

$= \frac{9(1-10^n)}{1-10}$

$= 10^n - 1$

3 $u_3 = 2 \qquad u_7 = \frac{1}{128}$

$u_1 r^2 = 2 \quad u_1 r^6 = \frac{1}{128}$

$\frac{u_1 r^6}{u_1 r^2} = \frac{\frac{1}{128}}{2}$

$\therefore\ r^4 = \frac{1}{256}$

$r = \frac{1}{4}$ or $\frac{-1}{4} \quad u_1 = 32$

$S_6 = \frac{32\left(1-\left(\frac{1}{4}\right)^6\right)}{1-\frac{1}{4}} = \frac{1365}{32} = 42.7$

or $S_6 = \frac{32\left(1-\left(\frac{-1}{4}\right)^6\right)}{1-\left(-\frac{1}{4}\right)} = \frac{819}{32} = 5.6$

4 **a** $u_1 = S_1 = \frac{3}{2} - 1 = \frac{1}{2}, u_2 = S_2 - S_1 = \left(\frac{3}{2}\right)^2 - \frac{3}{2} = \frac{3}{4},$

$u_3 = S_3 - S_2 = \left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^2 = \frac{9}{8}$

b $u_n = \left(\frac{3}{2}\right)^n - \left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^{n-1}\left(\frac{3}{2}-1\right)$

$= \frac{1}{2} \times \left(\frac{3}{2}\right)^{n-1}$

This is a GP with $u_1 = \frac{1}{2}$ and $r = \frac{3}{2}$

5 $P_n = a \times a\,r \times a\,r^2 \times \ldots \times a\,r^{n-1}$

$= a^n\,r^{1+2+\ldots+n-1}$

$= a^n\,r^{\frac{(n-1)}{2}n}$

Reciprocal sequence $= \frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}, \ldots, \frac{1}{ar^{n-1}}, \ldots$

i.e. a GP with $u_1 = \frac{1}{a}$, and common ratio $\frac{1}{r}$.

$$R_n = \frac{\frac{1}{a}\left(1-\frac{1}{r^n}\right)}{1-\frac{1}{r}} = \frac{1}{a} \times \frac{r^n-1}{r^n} \times \frac{r}{r-1} = \frac{1}{a}\,\frac{r^n-1}{(r-1)r^{n-1}}$$

$$\frac{S_n}{R_n} = \frac{a(1-r^n)}{1-r} \times \frac{a(r-1)r^{n-1}}{r^n-1} = a \times -1 \times a \times -1\;r^{n-1}$$

$$= a^2\,r^{n-1}$$

Hence $\left(\frac{S_n}{R_n}\right)^n = a^{2n}r^{n\,(n-1)}$

$= \left(a^n r^{\frac{(n-1)n}{2}}\right)^2$

$= P_n^{\,2}$ QED

6 $ar = 24$

$a\,r^2 = 12\,(P-1) \Rightarrow r = \frac{P-1}{2}$

But $|r| < |$ so $-1 < \frac{P-1}{2} < 1$ i.e. $-2 < P - 1 < 2$

$\Rightarrow -1 < P < 3$ (1)

Also $S_3 = 76$ so $\frac{48}{P-1} + 24 + 12\,(P-1) = 76$

$\Rightarrow 48 + 24\,(P-1) + 12\,(P-1)^2 = 76\,(P-1)$

$\Rightarrow 48 - 24 + 12P^2 + 12 = 76P - 76$

$\Rightarrow 12P^2 - 76P + 112 = 0$

$\Rightarrow 3P^2 - 19P + 28 = 0$

$(3P-7)\,(P-4) = 0$

$\Rightarrow P = \frac{7}{3}$ or 4

From convergence condition (1), $P = \frac{7}{3}$

Hence $r = \frac{\frac{4}{3}}{2} = \frac{2}{3}$

7 The lengths are a, ar, ar^2,

Where $a + ar + ar^2 = 2$ (1)

But $a\,r^2 = 2a$

so $r^2 = 2$ and $r = \pm\sqrt{2}$.

As a, $a\,r$, $a\,r^2$ are lengths, r must be positive so $r = \sqrt{2}$.

Substitute into (1) $\Rightarrow a\,(1 + \sqrt{2} + 2) = 2$

$\Rightarrow a = \frac{2}{3+\sqrt{2}} = \frac{2}{7}\,(3 - \sqrt{2})$ metres.

8 $1, \frac{x+1}{3}, \frac{(x+1)^2}{9}, \frac{(x+1)^3}{27}$

Convergent when $x = -1.5 = -\frac{3}{2}$

$$S_4 = 1\,\frac{\left(1-\frac{(x+1)^4}{3^4}\right)}{1-\frac{x+1}{3}}$$

$$= \frac{185}{216}$$

9 $\frac{1}{1-r} = \frac{(1-r^n)}{1-r} = k\,r^{n-1}$

$\Rightarrow 1 - (1 - r^n) = k\,r^{n-1}\,(1 - r)$

$\Rightarrow r^n = k\,r^{n-1}\,(1 - r)$

$\Rightarrow r = k\,(1 - r)$

$\Rightarrow (1 + k)r = k \Rightarrow r = \frac{k}{1+k}$

Hence $S = \frac{a}{1-\frac{k}{1+k}} = \frac{a(1+k)}{1+k-k}$

$= a\,(1 + k) = (k + 1)a$

$= (k + 1)u_1$

Exercise 1F

1 **a** $S = 4\,u_2\,\frac{u_1}{1-r} = 4\,u_1 r$

$1 = 4r(1 - r)$

$1 = 4r - 4r^2$

$4r^2 - 4r + 1 = 0$

$(2r - 1)^2 = 0$

$r = \frac{1}{2}$

b $u_1 = 32 \quad r = \frac{1}{2} \quad S_5 = \frac{32\left(1-\left(\frac{1}{2}\right)^5\right)}{1-\frac{1}{2}} = 62$

$S = \frac{32}{1-\frac{1}{2}} = 64$

percentage error $= \frac{2}{62} \times 100 = 3.23\%$

2 $r = 1.5 \quad S_5 = 52750$

$\frac{u_1(1-1.5^5)}{1-1.5} = 52750$

$u_1 = \$4000$

3 **a** $2 + 4 + 8 + 16 + 32 = 62$

b $S_n > 1000\,000$

$\frac{2(1-2^n)}{1-2} > 100\,0000$

$(2^n - 1) > 500\,000$

$2^n > 500001$

$n = 19$

4 **a** Let x = monthly repayment

Amount owing after 1 month

$= 1000 \times 1.01 - x$

Amount owing after 2 months

$= (1000 \times 1.01 - x) \times 1.01 - x$

$= 1000 \times 1.01^2 - 1.01x - x$

Amount owing after 3 months

$= (1000 \times 1.01^2 - 1.01x - x) \times 1.01 - x$

$= 1000 \times 1.01^3 - 1.01^2x - 1.01x - x$

Amount owing after 24 months

$= 1000 \times 1.01^{24} - 1.01^{23}x - 1.01^{22}x$
$- 1.01^{21}x \ldots . - 1.01x - x$

We require this to be zero

$\therefore x + 1.01x + 1.01^2x + \ldots + 1.01^{23}x$
$= 1000 \times 1.01^{24}$

$\frac{x(1-1.01^{24})}{1-1.01} = 1000 \times 1.01^{24}$

$x = \$47.07$

b Total to be paid $= 47.07 \times 24$
$= \$1130$

Exercise 1G

1 **a** Odd number + even number $= 2a + 1 + 2b$
$= 2(a + b) + 1,$
which is odd.

b Odd number × odd number $= (2m + 1)(2n + 1)$
$= 4mn + 2m + 2n + 1 = 2(m + n + 2mn) + 1,$
which is odd.

2 $\frac{1}{x-2} - \frac{2}{2x+5} = \frac{2x+5-2(x-2)}{(x-2)(2x+5)}$

$\therefore \frac{1}{x-2} - \frac{2}{2x+5} = \frac{9}{2x^2+x-10}$

3 $(a + b)^2 = c^2 + 4\left(\frac{ab}{2}\right)$

$a^2 + 2ab + b^2 = c^2 + 2ab$

$\therefore a^2 + b^2 = c^2$

4

3	4	3 × 4 + 4	16
7	8	7 × 8 + 8	64
–6	–5	–6 × –5 + –5	25
11	12	11 × 12 + 12	144
8	9	8 × 9 + 9	81

The product of two consecutive integers plus the larger of the two integers is equal to the square of the larger integer.

Proof: Let the two integers be n and $n + 1$

$n(n + 1) + (n + 1) = (n + 1)(n + 1) = (n + 1)^2$

Exercise 1H

1 $p(n)$: $S_n = \frac{u_1(1-r^n)}{1-r}$

Step 1: when $n = 1$, LHS $= S_1 = u_1$

RHS $= \frac{u_1(1-r)}{1-r} = u_1$

$\therefore p(1)$ is true

Step 2: assume $p(k)$ i.e., $S_k = \frac{u_1(1-r^k)}{1-r}$

Step 3: prove $p(k + 1)$ i.e., $S_{k+1} = \frac{u_1(1-r^{k+1})}{1-r}$

Proof: $S_{k+1} = S_k + u_{k+1}$

$= S_k + u_1r^k$

$= \frac{u_1(1-r^k)}{1-r} + u_1r^k$

$= \frac{u_1(1-r^k)+u_1r^k(1-r)}{1-r}$

$= \frac{u_1(1-r^k+r^k-r^{k+1})}{1-r}$

$\therefore S_{k+1} = \frac{u_1(1-r^{k+1})}{1-r}$

Since $p(1)$ is true and if $p(k)$ is true then $p(k + 1)$ is true, by the principle of mathematical induction, $p(n)$ is true

2 **a** $p(n)$: $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n + 1)(2n + 1)$

Step 1: when $n = 1$, LHS = 1

RHS $= \frac{1}{6}(2)(3) = 1$

$\therefore p(1)$ is true

Step 2: assume $p(k)$ i.e., $\sum_{r=1}^{k} r^2 = \frac{k}{6}(k + 1)(2k + 1)$

Step 3: prove $p(k + 1)$ i.e., $\sum_{r=1}^{k+1} r^2 = \frac{(k+1)}{6}(k + 2)(2k + 3)$

Proof: $\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k + 1)^2$

$= \frac{k}{6}(k + 1)(2k + 1) + (k + 1)^2$

$= \frac{(k+1)}{6}[k(2k + 1) + 6(k + 1)]$

$= \frac{(k+1)}{6}[2k^2 + 7k + 6]$

$\therefore \sum_{r=1}^{k+1} r^2 = \frac{(k+1)}{6}(k + 2)(2k + 3)$

Since $p(1)$ is true and if $p(k)$ is true then $p(k + 1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

b $p(n)$: $\sum_{r=1}^{n} 2^{r-1} = 2^n - 1$

Step 1: when $n = 1$, LHS $= 2^0 = 1$

RHS $= 2^1 - 1 = 1$

$\therefore p(1)$ is true

Step 2: assume $p(k)$ i.e., $\sum_{r=1}^{k} 2^{r-1} = 2k - 1$

Step 3: prove $p(k+1)$ i.e., $\sum_{r=1}^{k+1} 2^{r-1} = 2^{k+1} - 1$

Proof: $\sum_{r=1}^{k+1} 2^{r-1} = \sum_{r=1}^{k} 2^{r-1} + 2^k$

$= 2^k - 1 + 2^k = 2(2^k) - 1$

$\therefore \sum_{r=1}^{k+1} 2^{r-1} = 2^{k+1} - 1$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

c $p(n)$: $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2}{4}(n+1)^2$

Step 1: when $n = 1$, LHS $= 1^3 = 1$

RHS $= \frac{1}{4}(2)^2 = 1$

$\therefore p(1)$ is true

Step 2: assume $p(k)$ i.e., $1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{k^2}{4}(k+1)^2$

Step 3: prove $p(k+1)$ i.e.,

$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \frac{(k+1)^2}{4}(k+2)^2$

Proof: $1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3$

$= \frac{k^2}{4}(k+1)^2 + (k+1)^3 = \frac{(k+1)^2}{4}[k^2 + 4k + 4]$

$\therefore 1^3 + 2^3 + 3^3 + \dots + (k+1)^3 = \frac{(k+1)^2}{4}(k+2)^2$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

d $p(n)$: $\sum_{r=1}^{n} r(r+2) = \frac{n}{6}(n+1)(2n+7)$

Step 1: when $n = 1$, LHS $= 1(3) = 3$

RHS $= \frac{1}{6}(2)(9) = 3$

$\therefore p(1)$ is true

Step 2: assume $p(k)$ i.e.,

$\sum_{r=1}^{k} r(r+2) = \frac{k}{6}(k+1)(2k+7)$

Step 3: prove $p(k+1)$ i.e.,

$\sum_{r=1}^{k+1} r(r+2) = \frac{(k+1)}{6}(k+2)(2k+9)$

Proof: $\sum_{r=1}^{k+1} r(r+2) = \sum_{r=1}^{k} r(r+2) + (k+1)(k+3)$

$= \frac{k}{6}(k+1)(2k+7) + (k+1)(k+3)$

$= \frac{(k+1)}{6}[k(2k+7) + 6(k+3)]$

$= \frac{(k+1)}{6}(2k^2 + 13k + 18)$

$\therefore \sum_{r=1}^{k+1} r(r+2) = \frac{(k+1)}{6}(k+2)(2k+9)$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

Exercise 1I

1 $p(n)$: $7^n - 1 = 6A \quad (A \in \mathbb{Z})$

Step 1: when $n = 1$, $7^n - 1 = 7 - 1 = 6$

$\therefore p(1)$ is true

Step 2: assume $p(k)$ i.e.,

$7^k - 1 = 6A$

Step 3: prove $p(k+1)$ i.e.,

$7^{k+1} - 1 = 6B \quad (B \in \mathbb{Z})$

Proof: $7^{k+1} - 1 = 7(7^k) - 1$

$= 7(6A + 1) - 1$

$= 42A + 7 - 1$

$= 42A + 6$

$= 6(7A + 1)$

$\therefore 7^{k+1} - 1 = 6B$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

2 $p(n)$: $1 + 3 + 5 + 7 + \dots + (2n-1) = n^2$

Step 1: when $n = 1$, LHS $= 1$

RHS $= 1^2 = 1$

$\therefore p(1)$ is true

Step 2: assume $p(k)$ i.e.,

$1 + 3 + 5 + 7 + \dots + (2k-1) = k^2$

Step 3: prove $p(k+1)$ i.e.,

$1 + 3 + 5 + 7 + \dots + (2k-1) + (2k+1) = (k+1)^2$

Proof: $1 + 3 + 5 + 7 + \dots + (2k-1) + (2k+1)$

$= k^2 + (2k+1)$

$\therefore 1 + 3 + 5 + 7 + \dots + (2k+1) = (k+1)^2$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

3 $p(n)$: $9^n - 1 = 8A$, where $A \in \mathbb{Z}$

Step 1: when $n = 1$, $9^n - 1 = 8$ $p(1)$ is true

Step 2: Assume $p(k)$ i.e.

$9^k - 1 = 8A$

Step 3: prove $p(k+1)$ i.e.

$9^{k+1} - 1 = 8B \ (B \in \mathbb{Z})$

Proof: $9^{k+1} - 1 = 9 \times 9^k - 1$

$= 9(8A + 1) - 1$

$= 72A + 8$

$= 8(9A + 1)$

$\therefore 9^{k+1} - 1 = 8B$

Since $p(1)$ is true and if $p(k)$ is true then $p(k+1)$ is true, by induction, $p(n)$ is true

4 $p(n)$: $n^3 - n = 6A$, where $A \in \mathbb{Z}$

Step 1: when $n = 1$, $n^3 - n = 0 = 6 \times 0$

$\therefore$ $p(1)$ is true

Step 2: Assume $p(k)$ i.e.

$k^3 - k = 6A$

Step 3: prove $p(k + 1)$ i.e. $(k + 1)^3 - (k + 1) = 6B$ where $B \in \mathbb{Z}$

$$\begin{aligned}\text{Proof: } (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1\\ &= k^3 + 3k^2 + 2k\\ &= 6A + k + 3k^2 + 2k\\ &= 6A + 3(k^2 + k)\\ &= 6A + 3k(k+1)\end{aligned}$$

But $k(k + 1)$ is either odd × even or even × odd so is divisible by 2.

$\therefore$ $3k(k + 1)$ is divisible by 6.

$\therefore$ $(k + 1)^3 - (k + 1) = 6B$

$\therefore$ Since $p(1)$ is true and if $p(k)$ is true then $p(k + 1)$ is true, by induction $p(n)$ is true.

5 $p(n)$: $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$

Step 1: when $n = 1$, $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{1}{1\times 2} = \frac{1}{2}$

and $\dfrac{n}{n+1} = \dfrac{1}{2}$ $\therefore$ $p(1)$ is true

Step 2: assume $p(k)$ i.e. $\displaystyle\sum_{r=1}^{k}\frac{1}{r(r+1)} = \frac{k}{k+1}$

Step 3: prove $p(k + 1)$ i.e. $\displaystyle\sum_{r=1}^{k+1}\frac{1}{r(r+1)} = \frac{k+1}{k+2}$

$$\text{Proof: } \sum_{r=1}^{k+1}\frac{1}{r(r+1)} = \sum_{r=1}^{k}\frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Since $p(1)$ is true, and if $p(k)$ is true then $p(k + 1)$ is true, by induction $p(n)$ is true.

6 $p(n)$: $2^{n+2} + 3^{2n+1} = 7A$ where $A \in \mathbb{Z}$

Step 1: when $n = 1$, $2^{n+2} + 3^{2n+1} = 2^2 + 3^3$

$= 8 + 27 = 35 + 7 \times 5$

$\therefore$ $p(1)$ is true

Step 2: assume $p(k)$ i.e. $2^{k+2} + 3^{2k+1} = 7A$

Step 3: prove $p(k + 1)$ i.e. $2^{k+3} + 3^{2k+3} = 7B$ where $B \in \mathbb{Z}$

$$\begin{aligned}\text{Proof: } 2^{k+3} + 3^{2k+3} &= 2(7A - 3^{2k+1}) + 3^{2k+3}\\ &= 14A + 3^{2k+3} - 2 \times 32^{k+1}\\ &= 14A + 3^{2k+1}(9 - 2)\\ &= 14A + 3^{2k+1} \times 7\\ &= 7(2A + 3^{2k+1} = 7B\end{aligned}$$

Since $p(1)$ is true, and if $p(k)$ is true then $p(k + 1)$ is true, by induction, $p(n)$ is true.

7 $1, \frac{1}{3}, -\frac{1}{9}, -\frac{11}{27}, -\frac{49}{81}$

$p(n)$: $u_n = 3\left(\frac{2}{3}\right)^n - 1$

Step 1: When $n = 1$ $u_1 = 1$ and $3\left(\frac{2}{3}\right)^1 - 1 = 2 - 1 = 1$

$\therefore$ $p(1)$ is true

Step 2: Assume $p(k)$ i.e. $u_k = 3\left(\frac{2}{3}\right)^k - 1$

Step 3: Prove $p(k + 1)$ i.e. $u_{k+1} = 3\left(\frac{2}{3}\right)^{k+1} - 1$

$$\begin{aligned}\text{Proof: } u_{k+1} &= \frac{2u_k - 1}{3}\\ &= \frac{2\times 3\left(\frac{2}{3}\right)^k - 2 - 1}{3}\\ &= \frac{2\times 3\left(\frac{2}{3}\right)^k - 3}{3}\\ &= 2 \times \left(\frac{2}{3}\right)^k - 1\\ &= \frac{2}{3} \times 3 \times \left(\frac{2}{3}\right)^k - 1\\ &= 3\left(\frac{2}{3}\right)^{k+1} - 1\end{aligned}$$

Since $p(1)$ is true, and if $p(k)$ is true then $p(k + 1)$ is true, therefore by induction, $p(n)$ is true.

Exercise 1J

1 $8! - 7! = 8 \times 7! - 7! = 7 \times 7!$

$10! - 9! = 10 \times 9! - 9! = 9 \times 9!$

$5! - 4! = 5 \times 4! - 4! = 4 \times 4!$

$95! - 94! = 95 \times 94! - 94! = 94 \times 94!$

$(n + 1)! - n! = (n + 1)\, n! - n! = n \times n!$

2 **a** $\dfrac{4!}{6!} = \dfrac{1}{6\times 5} = \dfrac{1}{30}$

b $\dfrac{5!\times 3!}{6!} = \dfrac{3!}{6} = 1$

c $\dfrac{8!\times 6!}{5!} = 8! \times 6 = 241920$

3 **a** $\dfrac{n!+(n-1)!}{(n+1)!} = \dfrac{n(n-1)!+(n-1)!}{(n+1)n(n-1)!} = \dfrac{n+1}{(n+1)n} = \dfrac{1}{n}$

b $$\begin{aligned}\frac{n!-(n-1)!}{(n-2)!} &= \frac{n(n-1)(n-2)!-(n-1)(n-2)!}{(n-2)!}\\ &= n(n-1) - (n-1)\\ &= (n-1)(n-1)\\ &= (n-1)^2\end{aligned}$$

c $\dfrac{(n!)^2-1}{n!+1} = \dfrac{(n!-1)(n!+1)}{n!+1} = n!-1$

4 $$\begin{aligned}\frac{(2n+2)!\,(n!)^2}{[(n+1)!]^2\,(2n)!} &= \frac{(2n+2)(2n+1)(2n)!\,(n!)^2}{(n+1)^2\,(n!)^2\,(2n)!}\\ &= \frac{2(n+1)(2n+1)}{(n+1)^2}\\ &= \frac{2(2n+1)}{(n+1)}\end{aligned}$$

Exercise 1K

1 $26 \times 25 \times 24 = 15\,600$

2 **a** $12! = 479\,001\,600$

b $4! \times 3! \times 4! \times 2! \times 3! = 41\,472$

3 $\binom{8}{4} = 70$ weeks

4 **a** $\binom{20}{4} = 4845$

b $4845 - \binom{8}{4} - \binom{12}{4} = 4845 - 70 - 495 = 4280$

5 **a** $6 \times 7 \times 7 \times 4 = 1176$

b must end in 0 $\quad 6 \times 7 \times 7 \times 1 = 294$

c ending in 0 $\quad 6 \times 5 \times 4 \times 1 = 120$

ending in 2, 4 or 6 $\quad 5 \times 5 \times 4 \times 3 = 300$

$120 + 300 = 420$

6 $26^3 \times 10^3 = 17\,576\,000$

Exercise 1L

1 **a** $\binom{n}{r} = \frac{n!}{(n-r)!\,r!} \quad \binom{n}{n-r} = \frac{n!}{(n-(n-r))!\,(n-r)!} = \frac{n!}{r!(n-r)!}$

$$\therefore \binom{n}{r} = \binom{n}{n-r}$$

b $\binom{n+1}{r} = \frac{(n+1)!}{(n+1-r)!\,r!}$

$$\binom{n}{r} + \binom{n}{r-1} = \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$= \frac{(n-r+1)n! + rn!}{(n-r+1)!\,r!}$$

$$= \frac{n \times n! + n!}{(n-r+1)!\,r!}$$

$$= \frac{n!(n+1)}{(n-r+1)!\,r!}$$

$$= \frac{(n+1)!}{(n-r+1)!\,r!}$$

$$\therefore \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

2 **a** $(1 + 2x)^{11} = 1 + \binom{11}{1}(2x) + \binom{11}{2}(2x)^2 + \binom{11}{3}(2x)^3 + \ldots.$

$= 1 + 22x + 220x^2 + 1320x^3 + \ldots.$

b $(1 - 3x)^7 = 1 + \binom{7}{1}(-3x) + \binom{7}{2}(-3x)^2 + \binom{7}{3}(-3x)^3 + \ldots.$

$= 1 - 21x + 189x^2 - 945x^3 + \ldots.$

c $(2 + 5x)^5 = 2^5 + \binom{5}{1}2^4(5x) + \binom{5}{2}2^3(5x)^2 + \binom{5}{3}2^2(5x)^3 + \ldots.$

$= 32 + 400x + 2000x^2 + 5000x^3 + \ldots.$

d $\left(2 - \frac{x}{3}\right)^9 = 2^9 + \binom{9}{1}2^8\left(\frac{-x}{3}\right) + \binom{9}{2}2^7\left(\frac{-x}{3}\right)^2 + \binom{9}{3}2^6\left(\frac{-x}{3}\right)^3 + \ldots.$

$= 512 - 768x + 512x^2 - \frac{1792}{9}x^3 + \ldots.$

3 **a** $(1 - 4x)^7$ 4th term $= \binom{7}{3}(-4x)^3 = -2240x^3$

b $\left(1 - \frac{x}{2}\right)^{20}$ 3rd term $= \binom{20}{2}\left(\frac{-x}{2}\right)^2 = \frac{95}{2}x^2$

c $(2a - b)^8$ 4th term $= \binom{8}{3}(2a)^5(-b)^3 = -1792a^5b^3$

4 $\binom{12}{4}(2x)^8\left(\frac{1}{x^2}\right)^4 = 126720$

5 $(2 + \frac{x}{5})^5 = 2^5 + \binom{5}{1}2^4 \cdot \frac{x}{5} + \binom{5}{2}2^3 \cdot \frac{x^2}{25} + \binom{5}{3}2^2 \cdot \frac{x^3}{125} + \binom{5}{4}2 \cdot \frac{x^4}{625} + \frac{x^5}{3125}$

$= 32 + 16x + \frac{80x^2}{25} + \frac{40x^3}{125} + \frac{2x^4}{125} + \frac{x^5}{3125}$

$= 32 + 16x + \frac{16x^2}{5} + \frac{8x^3}{25} + \frac{2x^4}{125} + \frac{x^5}{3125}$

$(2.01)^5 = (2 + \frac{0.05}{5})^5 = 32 + 0.8 + 0.008 + 0.00004 + 0.0000001 + \ldots$

$= 32.80804$ (5 dp)

6 **a** $(\sqrt{2} - \sqrt{3})^4 = 4 - 4 \times 2\sqrt{2} \times \sqrt{3} + 6 \times 2 \times 3 - 4 \times \sqrt{2} \times 3\sqrt{3}$

$= 4 - 8\sqrt{6} + 36 - 12\sqrt{6} + 9$

$= 49 - 20\sqrt{6}$

b $(\sqrt{2} + \frac{1}{\sqrt{5}})^3 = 2\sqrt{2} + 3 \times 2 \times \frac{1}{\sqrt{5}} + 3\sqrt{2} \times \frac{1}{\sqrt{5}} + \frac{1}{5\sqrt{5}}$

$= \frac{13\sqrt{2}}{5} + \frac{31}{5\sqrt{5}} = \frac{13}{5}\sqrt{2} + \frac{31}{5}\sqrt{5}$

c $(1 + \sqrt{7})^5 - (1 - \sqrt{7})^5 = 2 \times 5 \times \sqrt{7} + 2 \times 10 \times (\sqrt{7})^3 + 2 \times (\sqrt{7})^5$

$= 10\sqrt{7} + 140\sqrt{7} + 98\sqrt{7}$

$= 248\sqrt{7}$

7 **a** $a^2 - b^2 = x^2 + 2xy + y^2 - (x^2 - 2xy + y^2)$

$= 4xy$

$= 4\frac{(a+b)}{2}, \frac{(a-b)}{2}$ (using $2x = a + b$ and $2y = a - b$)

$= (a + b)(a - b) = (a - b)(a + b)$

b $a^3 = x^3 + 3x^2 y + 3xy^2 + y^3$

$b^3 = x^3 - 3x^2 y + 3xy^2 - y^3 \Rightarrow a^3 - b^3 = 6x^2 y + 2y^3$

$= 2y(3x^2 + y^2) = (a - b)(3x^2 + y^2)$

$= (a - b)\left[\frac{3(a+b)^2}{2^2} + \left(\frac{a-b}{2}\right)^2\right]$

$= \frac{(a-b)}{4}[3a^2 + 6ab + 3b^2 + a^2 - 2ab + b^2]$

$= \frac{(a-b)}{4}(4a^2 + 4ab + 4b^2)$

$= (a - b)(a^2 + ab + b^2)$

c $a^4 = x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$

$b^4 = x^4 - 4x^3 y + 6x^2 y^2 - 4xy^3 + y^4$

$\Rightarrow a^4 - b^4 = 8x^3 y + 8xy^3 = 8xy(x^2 + y^2)$

$= 8\frac{(a+b)}{2}\frac{(a-b)}{2}(x^2 + y^2)$

$= 2(a - b)(a - b)\left[\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2\right]$

$= 2(a - b)(a + b)\left[\frac{a^2}{2} + \frac{b^2}{2}\right]$

$= (a - b)(a + b)(a^2 + b^2)$

d $(a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1})$

e Let $p(n)$ be $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})$

When $n = 1$, $a^1 - b^1 = a - b$ so $p(1)$ is true.

Assume $p(n)$ is true for $n = k$ i.e. $a^k - b^k = (a - b)(a^{k-1} + a^{k-2}b + \ldots + b)$

Prove $p(n)$ is true for $n = k + 1$:

$a^{k+1} - b^{k+1} = a \times a^k - b^{k+1}$

$= a[(a - b)(a^{k-1} + a^{k-2}b + \ldots + b^{k-1})] + a \times b^k - b^{k+1}$

$= (a - b)[a^k + a^{k-1}b + \ldots + ab^{k-1}] + ab^k - b^{k+1}$

$= (a - b)(a^k + a^{k-1}b + \ldots + ab^{k-1}) + (a - b)b^k + b^{k+1} - b^{k+1}$

$= (a - b)(a^k + a^{k-1}b + \ldots ab^{k-1} + b^k)$

$\therefore p(k + 1)$ is true.

So, since $p(1)$ is true and if $p(k)$ is true then $p(k + 1)$ is true, therefore by induction $p(n)$ is true.

Review exercise

1 $u_2 = 16$ | $S_3 = 84$

$u_1 r = 16$ | $u_1 + u_1 r + u_1 r^2 = 84$

$u_1 = \frac{16}{r}$ | $u_1(1 + r + r^2) = 84$

$\frac{16}{r}(1 + r + r^2) = 84$

$16 + 16r + 16r^2 = 84r$

$16r^2 - 68r + 16 = 0$

$4r^2 - 17r + 4 = 0$

$(4r - 1)(r - 4) = 0 \quad r = \frac{1}{4}$ or 4

if $r = \frac{1}{4}$, $u_1 = 64$ 64, 16, 4

if $r = 4$, $u_1 = 4$ 4, 16, 64

2 $1 + 3 + 4 + 6 + 7 + 9 + 10 + 12 + \ldots\ldots + 46$

$= (1 + 4 + 7 + \ldots\ldots + 46) + (3 + 6 + 9 + \ldots\ldots + 45)$

$= \frac{16}{2}(1 + 46) + \frac{15}{2}(3 + 45)$

$= 376 + 360 = 736$

3 $c - b = b - a \qquad \frac{b}{a} = \frac{a}{c} \qquad a + b + c = \frac{-9}{2} \qquad (3)$

$\therefore a + c = 2b \quad (1) \quad \therefore bc = a^2 \quad (2)$

substitute (1) in (3) $\quad 2b + b = \frac{-9}{2}$

$3b = \frac{-9}{2} \quad \therefore b = \frac{-3}{2}$

$a + c = -3 \qquad \frac{-3}{2}c = a^2$

$c = -3 - a \qquad \therefore \frac{-3}{2}(-3 - a) = a^2$

$9 + 3a = 2a^2$

$2a^2 - 3a - 9 = 0$

$(2a + 3)(a - 3) = 0$

$a = \frac{-3}{2}$ or 3

$a \neq \frac{-3}{2}$ since $a \neq b \quad \therefore a = 3, \quad c = -6$

The three numbers are $3, \frac{-3}{2}, -6$

4 1, 3, 7, 15, 31, 63

$p(n)$: $u_n = 2^n - 1$

Step 1: when $n = 1$, $u_1 = 1 = 2^1 - 1$

$\therefore p(1)$ is true.

Step 2: assume $p(k)$ i.e. $u_k = 2^k - 1$

Step 3: prove $p(k + 1)$ i.e. $u_{k+1} = 2^{k+1} - 1$

proof: $u_{k+1} = 2u_k + 1$

$= 2(2^k - 1) + 1$

$= 2^{k+1} - 2 + 1$

$= 2^{k+1} - 1$

Since $p(1)$ is true and if $p(k)$ is true then $p(k + 1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

5 $p(n)$: $3^{2n} - 8n - 1 = 64A$ $(A \in \mathbb{Z}, \in \mathbb{Z}^+)$

Step 1: when $n = 1$, $3^2 - 8 - 1 = 0$

$\therefore p(1)$ is true.

Step 2: assume $p(k)$ i.e. $3^{2k} - 8k - 1 = 64A$

Step 3: prove $p(k + 1)$ i.e. $3^{2(k+1)} - 8(k + 1) - 1 = 64B$ $(B \in \mathbb{Z})$

Proof: $3^{2(k+1)} - 8(k + 1) - 1$

$= 3^{2k}(3^2) - 8k - 9$

$= 9(64A + 8k + 1) - 8k - 9$

$= 576A + 72k + 9 - 8k - 9$

$= 576A + 64k$

$= 64(9A + k)$

$= 6B$

Since $p(1)$ is true and if $p(k)$ is true then $p(k + 1)$ is true, by the principal of mathematical induction, $p(n)$ is true.

6 **a** $\begin{pmatrix} n+1 \\ 4 \end{pmatrix} = \frac{(n+1)!}{(n-3)!4!}$ **b** $\begin{pmatrix} n-1 \\ 2 \end{pmatrix} = \frac{(n-1)!}{(n-3)!2!}$

c $\frac{(n+1)!}{(n-3)!4!} = \frac{6(n-1)!}{(n-3)!2!}$

$\frac{(n+1)n}{24} = 3$

$n^2 + n = 72$

$n^2 + n - 72 = 0$

$(n + 9)\,(n - 8) = 0$

$\therefore n = 8$ (n cannot be negative)

7 $(1 + x)^n = \begin{pmatrix} n \\ 0 \end{pmatrix} + \begin{pmatrix} n \\ 1 \end{pmatrix} x + \begin{pmatrix} n \\ 2 \end{pmatrix} x^2 + \ldots + \begin{pmatrix} n \\ r \end{pmatrix} x^r + \ldots + \begin{pmatrix} n \\ n \end{pmatrix} x^n$

a Let $x = 1$, $\begin{pmatrix} n \\ 0 \end{pmatrix} + \begin{pmatrix} n \\ 1 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix} + \ldots + \begin{pmatrix} n \\ r \end{pmatrix} + \ldots + \begin{pmatrix} n \\ n \end{pmatrix} = 2^n$

b Let $x = -1$, $\begin{pmatrix} n \\ 0 \end{pmatrix} - \begin{pmatrix} n \\ 1 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix} - \ldots + (-1)^r \begin{pmatrix} n \\ r \end{pmatrix} + \ldots$

$+ (-1)^n \begin{pmatrix} n \\ n \end{pmatrix} = 0$

Review exercise

1 **a** $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = a^2$ $\quad a^2 = \frac{1}{2}$ $\quad a = \frac{1}{\sqrt{2}}$

$\left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 = b^2$ $\quad b^2 = \frac{1}{4}$ $\quad b = \frac{1}{2}$

$\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = c^2$ $\quad c^2 = \frac{1}{8}$ $\quad c = \frac{1}{2\sqrt{2}}$

b The spiral consists of 1.5 of the sides of the first eight squares and one of the sides of the ninth square.

$\text{length} = 1.5\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \ldots.\left(\frac{1}{\sqrt{2}}\right)^7\right) + \left(\frac{1}{\sqrt{2}}\right)^8$

$= 1.5\left(\frac{1-\left(\frac{1}{\sqrt{2}}\right)^8}{1-\frac{1}{\sqrt{2}}}\right) + \frac{1}{16} = 4.86$

c $\text{length} = 1.5\left(\frac{1}{1-\frac{1}{\sqrt{2}}}\right) = 5.12$

d The spiral consists of 8 triangles

$\text{Area} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{4}\right)^2 + \ldots.\right)$ to 8 terms

$= \frac{1}{2}\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots.\right)$ to 8 terms

$\text{Area} = \frac{1}{2}\left(\frac{\frac{1}{4}\left(1-\left(\frac{1}{2}\right)^8\right)}{1-\frac{1}{2}}\right) = 0.249$

e $\text{Area} = \frac{1}{2}\left(\frac{\frac{1}{4}}{1-\frac{1}{2}}\right) = 0.25$

2 **a** $\frac{14!}{3!\,2!\,2!\,2!\,2!} = 908\,107\,200$

b Consider 5 digit and 6 digit numbers ending in 0 or 5.

5 digit numbers:

$4 \times 6 \times 6 \times 6 \times 2 = 1728$

6 digit numbers:

$5 \times 6 \times 6 \times 6 \times 6 \times 2 = 12\,960$

$1728 + 12\,960 = 14\,688$

c $4! \times (2!)^4 = 384$

3

M	***W***
2	3
1	4

$\begin{pmatrix} 6 \\ 2 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

$= 15 \times 4 + 6 \times 1$

$= 66$

4 $\begin{pmatrix} 8 \\ 6 \end{pmatrix} = \left(x^3\right)^2\left(-\frac{3}{x}\right)^6 = 20412$

5 Coefficients are $\begin{pmatrix} n \\ r-1 \end{pmatrix}, \begin{pmatrix} n \\ r \end{pmatrix}, \begin{pmatrix} n \\ r+1 \end{pmatrix}$

$$\frac{n!}{(n-r-1)!\,(r-1)!} - \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r)!\,r!} - \frac{n!}{(n-r+1)!\,(r-1)!}$$

Divide by $n!$ and multiply by $(r + 1)!\,(n - r + 1)!$

$(n - r + 1)\,(n - r) - (r + 1)$

$(n - r + 1) = (r + 1)$

$(n - r + 1) - (r + 1)r$

$(n - r + 1)\,(n - r) - 2$

$(r + 1)\,(n - r + 1) +$

$(r + 1)r = 0$

$n^2 - rn - rn + r^2 + n \ - r - 2rn + 2r^2 - 2r - 2n +$

$2r - 2 + r^2 + r = 0$

$n^2 - 4rn + 4r^2 - n - 2 = 0$

$n^2 + 4r^2 - 2 - n\,(4r + 1) = 0$

$n = 14, \quad 196 + 4r^2 - 2 - 14(4r + 1) = 0$

$4r^2 - 56r + 180 = 0$

$r^2 - 14r + 45 = 0$

$(r - 5)\,(r - 9) = 0$

$r = 5$ or 9

The coefficients are $\begin{pmatrix} 14 \\ 4 \end{pmatrix}, \begin{pmatrix} 14 \\ 5 \end{pmatrix}, \begin{pmatrix} 14 \\ 6 \end{pmatrix}$ or $\begin{pmatrix} 14 \\ 8 \end{pmatrix}, \begin{pmatrix} 14 \\ 9 \end{pmatrix}, \begin{pmatrix} 14 \\ 10 \end{pmatrix}$

Both sets give 1001, 2002, 3003.

2 Mathematics as a language

Answers

Skills check

1 $y = x^2 - 3x - 1$

$y = \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$

Vertex is $\left(\frac{3}{2}, \frac{-13}{4}\right)$ Axis of symmetry is $x = \frac{3}{2}$

2 **a** $3x + 4 = 0,\ x = -\frac{4}{3}$

b $3x^2 - 2x - 1 = 0$

$(3x + 1)(x - 1) = 0$

$x = -\frac{1}{3}$ or 1

3 **a** $y - 3 = \sqrt{x-2}$

$\therefore x - 2 = (y - 3)^2$

$\therefore x = (y - 3)^2 + 2$

b $y = \frac{2x-1}{3x+2}$

$3xy + 2y = 2x - 1$

$2x - 3xy = 2y + 1$

$x(2 - 3y) = 2y + 1$

$x = \frac{2y+1}{2-3y}$

Exercise 2A

1 **a** function, domain = $\{0, 1, 2, 3\}$,
range = $\{-1, 1, 2, 3\}$

b function, domain = $\{-3, -2, -1, 0\}$,
range = $\{0\}$

c not a function

d not a function

2 **a** function, domain = $\{x \mid -3 \le x \le 3\}$,
range = $\{y \mid 0 \le y \le 3\}$

b not a function

c not a function

d function, domain = $\{x \mid x \ge -1\}$,
range = $\{y \mid y \ge 0\}$

Exercise 2B

1 $y^2 = x \Rightarrow y = \pm\sqrt{x}$
one value of x gives 2 values of y
eg. if $x = 4$, $y \pm 2$ $\therefore$ not a function.
$y = \sqrt{x}$, $\sqrt{x}$ is the positive square root of x
$\therefore$ each value of x gives just one value of y

2 **a** $y = x^2 - 4x + 2$ domain is $x \in \mathbb{R}$
$y = (x - 2)^2 - 2$ range = $\{y \mid y \ge -2\}$

b $y = -(x + 2)^2 - 3$ domain is $x \in \mathbb{R}$
range = $\{y \mid y \le -3\}$

c $y = \sqrt{x+2}$ $x + 2 \ge 0$ domain = $\{x \mid x \ge -2\}$
$x \ge -2$ range = $\{y \mid y \ge 0\}$

d $y = \sqrt{3-x}$ $3 - x \ge 0$ domain = $\{x \mid x \le 3\}$
$x \le 3$ range = $\{y \mid y \ge 0\}$

e $y = -3x^2 + 6x - 1$ domain is $x \in \mathbb{R}$
$= -3(x^2 - 2x) - 1$
$= -3[(x-1)^2 - 1] - 1$
$= -3(x-1)^2 + 2$ range = $\{y \mid y \le 2\}$

f $y = \sqrt{4-2x}$ $4 - 2x \ge 0$ domain = $\{x \mid x \le 2\}$
$x \le 2$ range = $\{y \mid y \ge 0\}$

Exercise 2C

1 $y = -|x|$ domain = $\{x \mid x \in \mathbb{R}\}$
range = $\{y \mid y \le 0\}$

2 $y = |2x + 1|$ domain = $\{x \mid x \in \mathbb{R}\}$
range = $\{y \mid y \ge 0\}$

3 $y = -|2x + 1|$ domain = $\{x \mid x \in \mathbb{R}\}$
range = $\{y \mid y \le 0\}$

4 $y = 2|x - 1|$ domain = $\{x \mid x \in \mathbb{R}\}$
range = $\{y \mid y \ge 0\}$

5 $y = -\frac{1}{2}|3x + 2|$ domain = $\{x \mid x \in \mathbb{R}\}$
range = $\{y \mid y \le 0\}$

6 $y = |x + 4| - 2$ domain = $\{x \mid x \in \mathbb{R}\}$
range = $\{y \mid y \ge -2\}$

7 $y = -2|x - 1| + 1$ domain = $\{x \mid x \in \mathbb{R}\}$
range = $\{y \mid y \le 1\}$

8 $y = 3|1 - 2x| - 2$ domain = $\{x \mid x \in \mathbb{R}\}$
range = $\{y \mid y \ge -2\}$

Exercise 2D

1 $y = \frac{1}{3x+2}$ domain = $\left\{x \mid x \ne -\frac{2}{3}\right\}$,
range = $\{y \mid y \ne 0\}$

2 $y = -\frac{1}{2-x}$ domain = $\{x \mid x \ne 2\}$,
range = $\{y \mid y \ne 0\}$

3 $y = \frac{3}{3-x}$ domain = $\{x \mid x \ne 3\}$,
range = $\{y \mid y \ne 0\}$

4 $y=-\frac{5}{6x+3}$ domain $=\left\{x\,|\,x\neq -\frac{1}{2}\right\}$,
range $=\{y\,|\,y\neq 0\}$

5 $y=\frac{1+2x}{1-2x}$ domain $=\left\{x\,|\,x\neq \frac{1}{2}\right\}$,
range $=\{y\,|\,y\neq -1\}$

6 $y=-\frac{2-3x}{1+x}$ domain $=\{x\,|\,x\neq -1\}$,
range $=\{y\,|\,y\neq 3\}$

Exercise 2E

1 $y=\frac{1}{|x+1|}$ domain $=\{x\,|\,x\neq -1\}$, range $=\{y\,|\,y>0\}$

2 $y=\frac{-2}{|x-1|}$ domain $=\{x\,|\,x\neq 1\}$, range $=\{y\,|\,y<0\}$

3 $y=\frac{x}{|x|}$ domain $=\{x\,|\,x\neq 0\}$
If $x>0,\ y=\frac{x}{x}=1$
If $x<0,\ y=\frac{x}{-x}=-1$
$\therefore$ range $=\{-1,1\}$

4 $y=\frac{-2}{\sqrt{1-x}}$ $1-x>0 \quad \therefore x<1$
domain $=\{x\,|\,x<1\}$, range $=\{y\,|\,y<0\}$

5 **a** For f to be real, $\sqrt{\frac{1}{x^2}-2}>0$

$$\Rightarrow \frac{1}{x^2}-2>0$$

$$\Rightarrow \frac{1}{x^2}-2>0$$

$$\therefore\ x^2<\frac{1}{2}$$

so domain$=\left\{x|-\frac{1}{\sqrt{2}}<x\,\frac{1}{\sqrt{2}},\ x\neq 0\right\}$

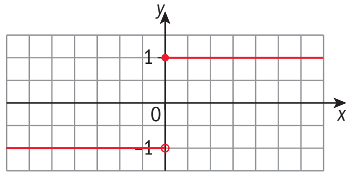
b Range $=\{y\,|\,y>0\}$

Exercise 2F

1 $y=\begin{cases}1, & x\geq 0\\ -1, & x<0\end{cases}$

a $f(-3)=-1 \quad f(0)=1 \quad f(\pi)=1 \quad f(4)=1$

b

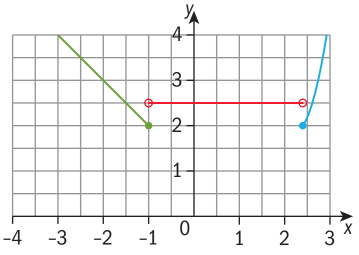


c domain $=\{x\,|\,x\in\mathbb{R}\}$ range $=\{-1,1\}$

2 $y=\begin{cases}1-x, & x\leq -1\\ 2.5, & -1<x<\sqrt{6}\\ x^2-4, & x\geq\sqrt{6}\end{cases}$

a $f(-3)=4 \quad f(0)=2.5 \quad f(\sqrt{6})=2 \quad f(3)=5$

b

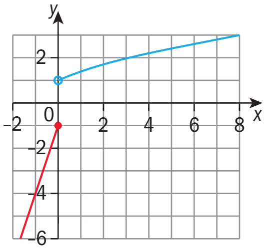


c domain $=\{x\,|\,x\in\mathbb{R}\}$ range $=\{y\,|\,y\geq 2\}$

3 $f(x)=\begin{cases}3x-1, & x\leq 0\\ \sqrt{x+1}, & x>0\end{cases}$

a $f(-1)=-4 \quad f(0)=-1 \quad f(1)=\sqrt{2} \quad f(8)=3$

b

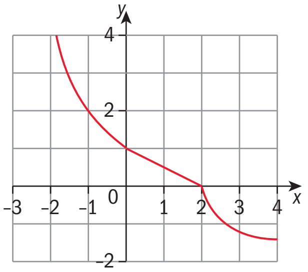


c domain $=\{x\,|\,x\in\mathbb{R}\}$
range $=\{y\,|\,y\leq -1 \text{ or } y>1\}$

4 $g(x)=\begin{cases}x^2+1, & x\leq 0\\ -\frac{1}{2}x+1, & 0<x\leq 2\\ -\sqrt{x-2}, & x>2\end{cases}$

a $f(-2)=5 \quad f(1)=\frac{1}{2} \quad f(2)=0 \quad f(3)=-1$

b



c domain $=\{x\,|\,x\in\mathbb{R}\}$ range $=\{y\,|\,y\in\mathbb{R}\}$

Exercise 2G

1 $f(x)=4-x^2$ **a** many-to-one
b $f(-x)=4-(-x^2)=4-x^2=f(x)$ $\therefore$ even

2 $g(x)=x^3+3x$ **a** one-to-one
b $g(-x)=(-x)^3+3(-x)=-x^3-3x=-g(x)$
$\therefore$ odd

3 $h(x)=\frac{-3}{2x}$ **a** one-to-one
b $h(-x)=\frac{-3}{2(-x)}=\frac{3}{2x}=-h(x)$ $\therefore$ odd

4 $p(x)=x^3+4x+1$ **a** one-to-one
b $p(-x)=(-x)^3+4(-x)+1=-x^3-4x+1$
$\neq p(x)$ or $-p(x)$ $\therefore$ neither odd nor even

5 $r(x)=\begin{cases}-1 & 0\leq x<\pi\\ 1 & \pi\leq x<2\pi\\ -1 & 2\pi<x<3\pi\end{cases}$ **a** many-to-one
b If $0\leq x<3\pi$, $r(-x)$ is not defined
$\therefore$ neither even nor odd

6 $q(x)=2x^3-4x$ **a** many-to-one
b $q(-x)=2(-x)^3-4(-x)=-2x^3+4x=-q(x)$
$\therefore$ odd

7 $w(x) = x - 2x^3 + x^5$ **a** many-to-one

b $w(-x) = -x - 2(-x)^3 + (-x)^5 = -x + 2x^3 - x^5$
$= -w(x)$

$\therefore$ odd

8 $t(x) = 4x^4 - x$ **a** many-to-one

b $t(-x) = 4(-x)^4 - (-x) = 4x^4 + x \neq t(x)$ or $-t(x)$

$\therefore$ neither even nor odd

9 $f(x) = 0$ is both even and odd

Exercise 2H

1 $f(x) = 2x \quad g(x) = \sqrt{x}$

domain of f is all real numbers

domain of g is all non-negative real numbers

a $2g(x) - f(x)$ domain is all non-negative real numbers

b $f(x) \cdot g(x)$ domain is all non-negative real numbers

c $\left(\frac{g}{f}\right)(x)$ domain is all positive real numbers

2 $f(x) = |x + 1| \quad g(x) = \sqrt{x^2 - 4}$

domain of f is all real numbers

domain of $g = \{x \mid x \leq -2, x \geq 2\}$

domain of $\left(\frac{f}{g}\right)(x) = \{x \mid x < -2, x > 2\}$

3 $f(x) = x^2 + 2x - 1 \qquad g(x) = 1 - 2x - 3x^2$

a $f(g(0)) = f(1) = 2$

b $g(f(-1)) = g(-2) = -7$

c $f(f(0)) = f(-1) = -2$

d $g(g(x)) = g(1 - 2x - 3x^2)$
$= 1 - 2(1 - 2x - 3x^2) - 3(1 - 2x - 3x^2)^2$
$= 1 - 2 + 4x + 6x^2 - 3(1 - 2x - 3x^2 - 2x + 4x^2$
$+ 6x^3 - 3x^2 + 6x^3 + 9x^4)$
$= -1 + 4x + 6x^2 - 3 + 12x + 6x^2 - 36x^3 - 27x^4)$
$= -4 + 16x + 12x^2 - 36x^2 - 27x^4$

4 $f(x) = 1 - 2x \quad g(x) = x^2 - 1 \quad h(x) = \sqrt{2x+4}$

a, b **i** $f(g(x)) = f(x^2 - 1)$
$= 1 - 2(x^2 - 1) = 3 - 2x^2$
domain $= \{x \mid x \in \mathbb{R}\}$ range $= \{y \mid y \leq 3\}$

ii $g(h(x)) = g\left(\sqrt{2x+4}\right) = 2x + 4 - 1$
$= 2x + 3$ domain
$=$ domain of $h = \{x \mid x \geq -2\}$ range
$= \{y \mid y \geq -1\}$

iii $f(h(x)) = f\left(\sqrt{2x+4}\right) = 1 - 2\sqrt{2x+4}$
domain $= \{x \mid x \geq -2\}$ range $= \{y \mid y \leq 1\}$

iv $h(g(x)) = h(x^2 - 1) = \sqrt{2(x^2 - 1) + 4}$
$= \sqrt{2x^2 + 2}$
domain $= \{x \mid x \in \mathbb{R}\}$ range $= \{y \mid y \geq \sqrt{2}\}$

d $f(g(1)) = 3 - 2(1)^2 = 1$
$h(f(g(1))) = h(1) = \sqrt{6}$

5 e.g. $f(x) = x - 2 \qquad g(x) = x^2$

6 e.g. $g(x) = 2x - 3 \qquad h(x) = \sqrt{x}$

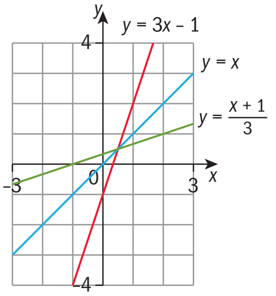
Exercise 2I

1 $y = 3x - 1 \qquad x = 3y - 1$

$\therefore \; y = \frac{x+1}{3} \qquad f^{-1}(x) = \frac{x+1}{3}$

$f(f^{-1}(x)) = 3\left(\frac{x+1}{3}\right) - 1 = x + 1 - 1 = x$

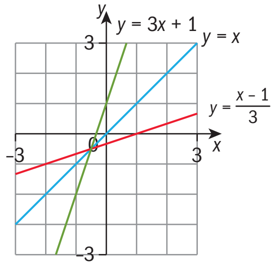
$f^{-1}(f(x)) = \frac{(3x-1)+1}{3} = \frac{3x}{3} = x$



2 $y = \frac{x-1}{3}$

$x = \frac{y-1}{3} \qquad \therefore \; y = 3x + 1 \;\; f^{-1}(x) = 3x + 1$

$f(f^{-1}(x)) = \frac{(3x+1)-1}{3} = \frac{3x}{3} = x$



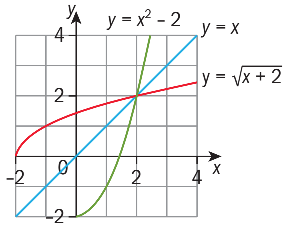
3 $y = x^2 - 2, x \geq 0$

$x = y^2 - 2 \;\; \therefore y = \sqrt{x+2}, x \geq -2$

$f^{-1}(x) = \sqrt{x+2}, x \geq -2$

$f(f^{-1}(x)) = \left(\sqrt{x+2}\right)^2 - 2 = x + 2 - 2 = x$

$f^{-1}(f(x)) = \sqrt{x^2 - 2 + 2} = \sqrt{x^2} = x$



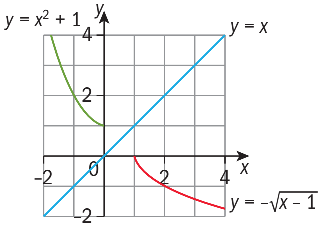
4 $y = x^2 + 1 \qquad x \leq 0$

$x = y^2 + 1$

$\therefore \; y = -\sqrt{x-1}, x \geq 1 \qquad f^{-1}(x) = -\sqrt{x-1}, x \geq 1$

$f(f^{-1}(x)) = \left(-\sqrt{x+1}\right)^2 + 1 = x - 1 + 1 = x$

$f^{-1}(f(x)) = -\sqrt{x^2 + 1 - 1} = -\sqrt{x^2} = x$



5 $y = x^2 + 4x - 1 \qquad x \geq -1$

$x = y^2 + 4y - 1$

$y^2 + 4y = x + 1$

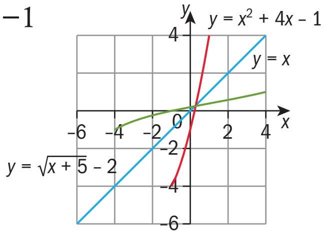
$y^2 + 4y + 4 = x + 5$

$(y + 2)^2 = x + 5$

$y + 2 = \sqrt{x+5}$

$y = \sqrt{x+5} - 2$

$f^{-1}(x) = \sqrt{x+5} - 2, \quad x \geq -4$ (since range of $f(x)$ is $y \geq -4$)



6 $y = 1 - 2x \qquad x = 1 - 2y$

$2y = 1 - x$

$y = \frac{1-x}{2}$

$\therefore\ y = 1 - 2x$ is not its own inverse

7 $f(x) = 3x \qquad g(x) = 2x + 1$

$f^{-1}(x) = \frac{x}{3} \qquad g^{-1}(x) = \frac{x-1}{2}$

$g \circ f(x) = g(3x) = 2(3x) + 1 = 6x + 1$

$(g \circ f)^{-1}(x) = \frac{x-1}{6} \quad f^{-1} \circ g^{-1}(x) = f^{-1}\left(\frac{x-1}{2}\right) = \frac{x-1}{6}$

$\therefore\ f^{-1} \circ g^{-1}(x) = (g \circ f)^{-1}(x)$

8 $y = \frac{2x+1}{x-1}$

$x = \frac{2y+1}{y-1}$

$y\,x - x = 2y + 1$

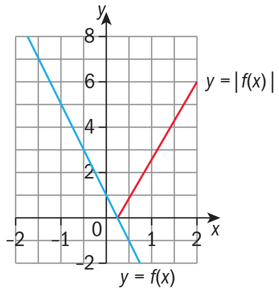
$y\,(x - 2) = 1 + x$

$y = \frac{1+x}{x-2}$

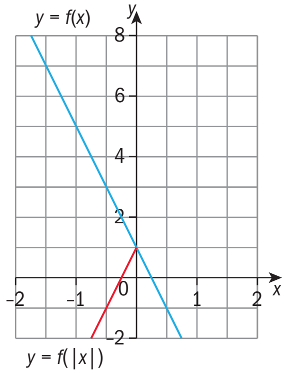
$\therefore\ f^{-1}(x) = \frac{1+x}{x-2}$, domain = $\{x \mid x \neq 2\}$

Exercise 2J

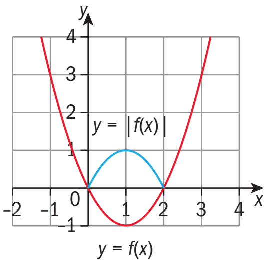
1 a



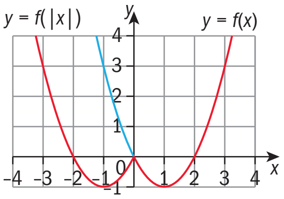
b



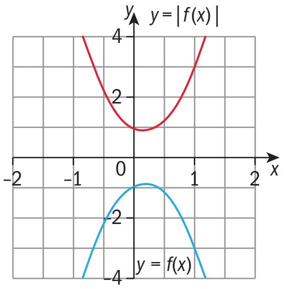
2 a



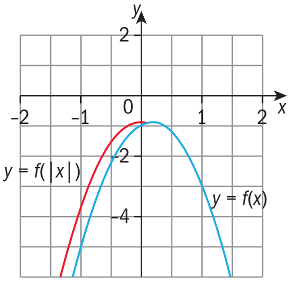
b



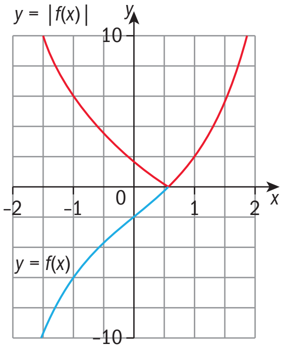
3 a



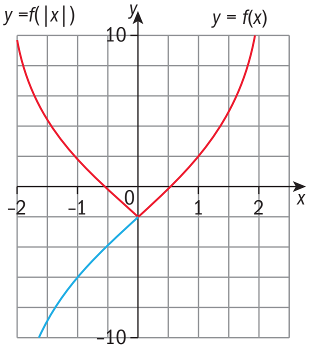
b



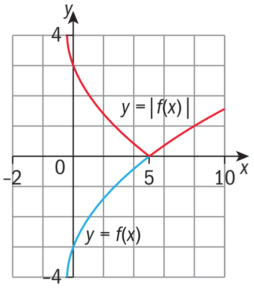
4 a



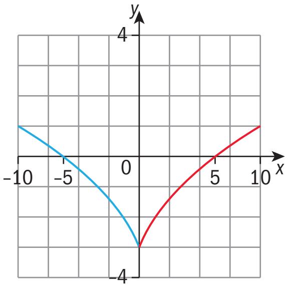
b



5 **a**

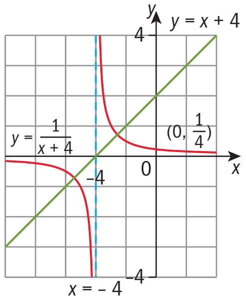


b



Exercise 2K

1



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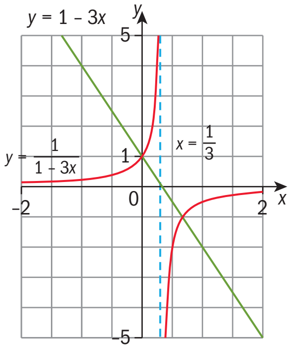

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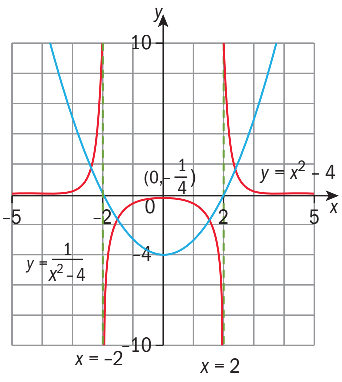

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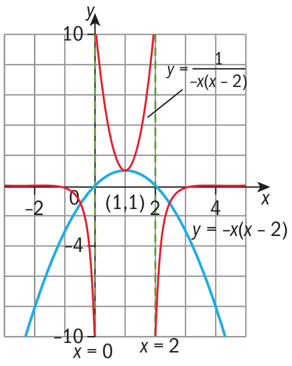

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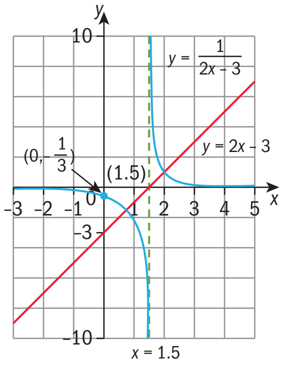
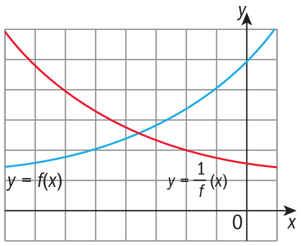
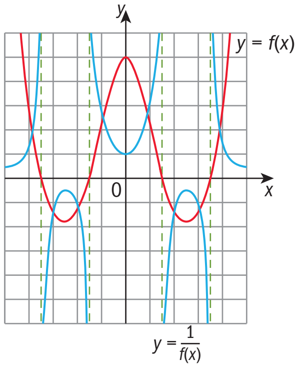


6 **a**



b



Exercise 2L

1 **a** Translation $\begin{pmatrix}3\\2\end{pmatrix}$, i.e. $g(x) = f(x-3)+2$

b Translation $\begin{pmatrix}-2\\-1\end{pmatrix}$, i.e. $g(x) = f(x+2)-1$

c Reflection in the x-axis and translation $\begin{pmatrix}0\\-1\end{pmatrix}$, i.e. $g(x) = -f(x)-1$

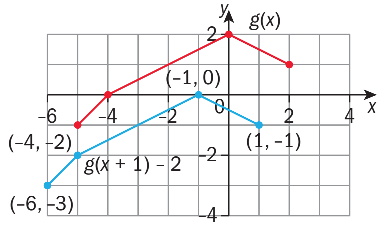
d Horizontal compression by a factor of $\frac{1}{2}$, i.e. $g(x) = f(2x)$

e Reflection in the x-axis and vertical stretch of 2, i.e. $g(x) = -2f(x)$

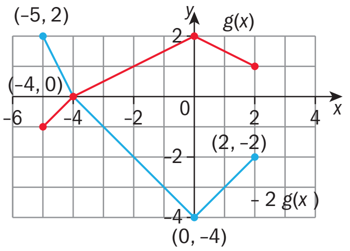
f Reflection in the y-axis and horizontal stretch of 2, i.e. $g(x) = f\left(-\frac{1}{2}x\right)$

2 $g(x) = h(-(x-3))$ or $g(x) = h(-x+3)$

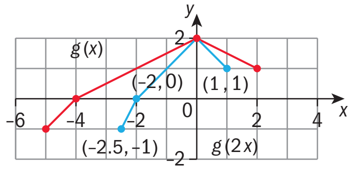
3 a



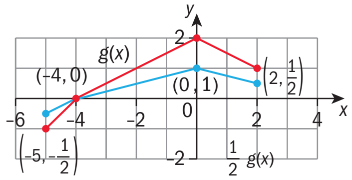
b



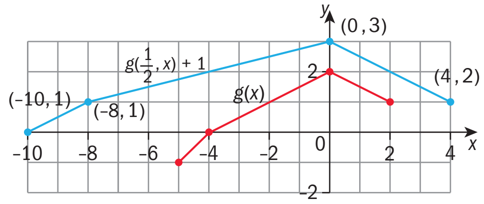
c



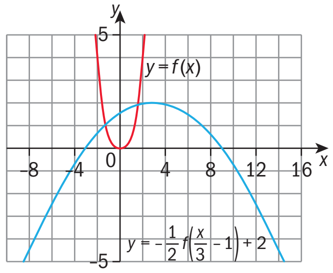
d



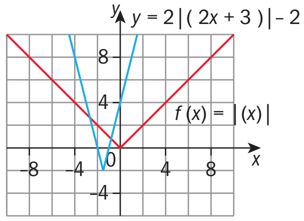
e



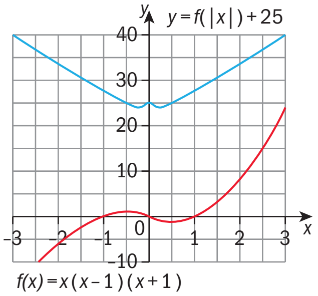
4 a



b



c



5 $y = \frac{2}{3(x+2)} + 3$ $y = \frac{2+9(x+2)}{3(x+2)}$ $y = \frac{9x+20}{3(x+2)}$

Domain = $\{x \mid x \neq -2\}$, Range = $\{y \mid y \neq 3\}$

6 a $x = -\frac{1}{2}$, $y = 2$

b When $x = 0$, $y = \frac{5}{1} = 5$

$\Rightarrow$ intercept at $(0, 5)$

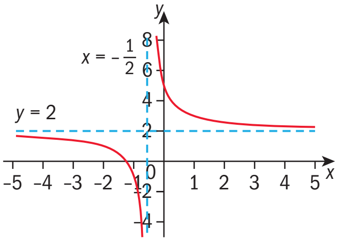
When $y = 0$, $4x + 5 = 0$

$4x = -5$

$x = -\frac{5}{4}$

$\Rightarrow$ intercept at $\left(-\frac{5}{4}, 0\right)$

c



d $y = \frac{2(2x+1)+3}{2x+1} = 2 + \frac{3}{2x+1}$

$\therefore$ if $g(x) = \frac{1}{x}$, then $f(x) = 3\,g(2x+1) + 2$

i.e. vertical stretch of 3, then horizontal stretch of $\frac{1}{2}$, then translation $\begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$.

Review exercise

1 a function domain = $\{x \mid x \in \mathbb{R}\}$

range = $\{y \mid y \leq 2\}$

b not a function

c not a function

d function domain = $\{x \mid x \in \mathbb{R}, x \neq \pm 1\}$

range = $\{y \mid y \leq -0.25 \text{ or } y > 0\}$

2 $f(g(h(3))) = f(g(4)) = f(2) = 3$

$h^{-1}(g^{-1}(f^{-1}(3))) = h^{-1}(g^{-1}(2)) = h^{-1}(4) = 3$

3 $f \circ f(x) = f\left(\frac{1}{x-1}\right) = \frac{1}{\frac{1}{x-1} - 1} = \frac{x-1}{1-(x-1)} = \frac{x-1}{2-x}$

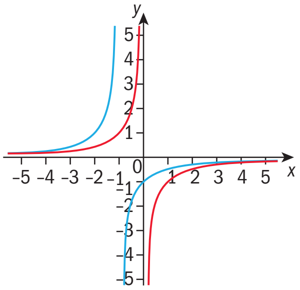
$y = \frac{x-1}{2-x}$ $x = \frac{y-1}{2-y}$ $2x - yx = y - 1$

$y + yx = 2x + 1$ $y(1 + x) = 2x + 1$ $y = \frac{2x+1}{x+1}$

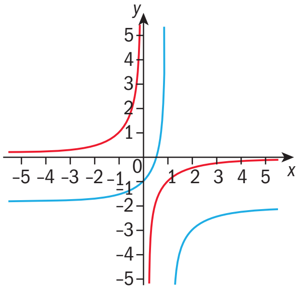
$\therefore \ (f \circ f)^{-1}(x) = \frac{2x+1}{x+1}$

4 $(-5, -2) \rightarrow (-6, 1)$ $(-4, 0) \rightarrow (-5, -3)$
$(-3, 2) \rightarrow (-4, -7)$ $(-1, -1) \rightarrow (-2, -1)$
$(3, -3) \rightarrow (2, 3)$ $(8, 2) \rightarrow (7, -7)$

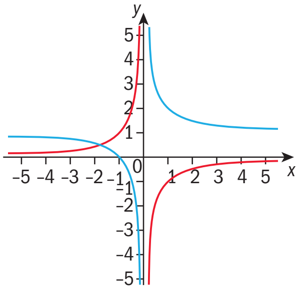
5 **a**



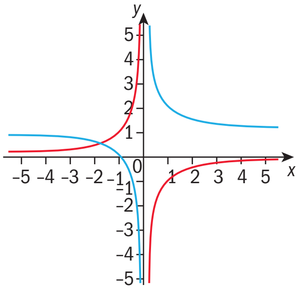
b



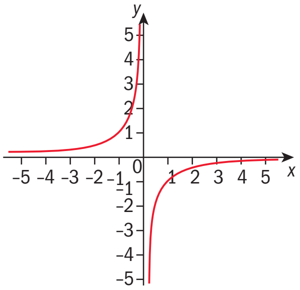
c



d



e



6 **a** Reflection in x-axis $\Rightarrow g(x) = -f(x)$

b Reflection in y-axis $\Rightarrow g(x) = f(-x)$

c Horizontal shift of 3 units to the left, vertical shift of 1 unit down $\Rightarrow g(x) = f(x + 3) -1$

d Reflection in the horizontal line $y = 1$
$\Rightarrow g(x) =-f(x) + 1$

e $g(x) = \frac{1}{f(x)}$

f Compression by factor of $\frac{1}{2}$ and reflection in x-axis $\Rightarrow g(x) = -f(2x)$

g Reflection in y-axis and vertical shift of 2 units upwards $\Rightarrow g(x) = f(-x) + 2$

7 $f(x)$ is odd, so $f(-x) = -f(x)$
Let $x = 0$, then $f(0) = -f(0)$
i.e. $2f(0) = 0$
$\therefore f(0) = 0$
$\therefore f(x)$ passes through $(0, 0)$

Review exercise

1 **a** $f(g(x)) = f(x^2) = 3x^2 - 1$

b $h(g(x)) = h(x^2) = \frac{1}{x^2+2}$

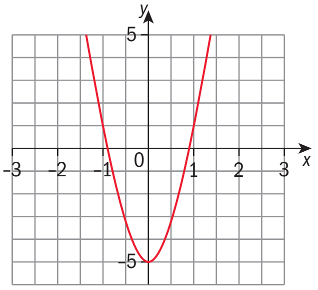
c $g^{-1}(x) = \sqrt{x}$ $g^{-1}(f(x))$

$= g^{-1}(3x - 1) = \sqrt{3x-1}$

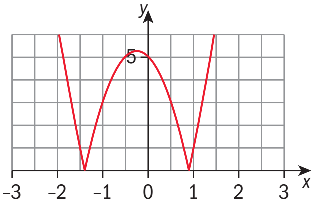
d Find h^{-1}: $x = \frac{1}{y+2}$ $\therefore y + 2 = \frac{1}{x} \therefore y = \frac{1}{x} - 2$
$f(h^{-1}(x)) = f(\frac{1}{x} - 2) = 3(\frac{1}{x} - 2) - 1 = \frac{3}{x} - 7$
$= \frac{3-7x}{x}$

2 $f(x) = \frac{x-1}{x+3}$ $g(x) = x^2$

3 $f(x) = |x|$ $g(x) = 4x^2 + 2x - 5$
$g(f(x)) = 4|x|^2 + 2|x| - 5$



$f(g(x)) = |4x^2 + 2x - 5|$



4 Vertical stretch of $\frac{9}{5}$ followed by translation $\begin{pmatrix} 0 \\ 32 \end{pmatrix}$
$x = \frac{5}{9}x - \frac{160}{9}$
$9x = 5x - 160$
$4x = -160$ and $\therefore x = -40$

5 $g(x) = \frac{3}{2(x+1)} - 2$
$= \frac{3-4(x+1)}{2(x+1)} = \frac{-1-4x}{2(x+1)}$

3 The long journey of mathematics

Answers

Skills check

1 **a** $x^2 + 2x - 3 = 0$

$(x + 3)(x - 1) = 0$

$x = -3$ or 1

b $x^2 - 11x + 10 = 0$

$(x - 10)(x - 1) = 0$

$x = 1$ or 10

c $2x^2 + x - 3 = 0$

$(2x + 3)(x - 1) = 0$

$x = -\frac{3}{2}$ or 1

2 **a** $f(x) + g(x) = 2x^3 - 3$

b $2h(x) - 4g(x) + 5f(x) = 6x^4 - 4x^2 - 10 - 8x^3 + 4x^2 - 12x + 16 + 5x^2 - 15x + 5$

$= 6x^4 - 8x^3 + 5x^2 - 27x + 11$

c $\frac{1}{2}h(x) - \frac{2}{5}g(x) = \frac{3}{2}x^4 - x^2 - \frac{5}{2} - \frac{4}{5}x^3 + \frac{2}{5}x^2 - \frac{6}{5}x + \frac{8}{5}$

$= \frac{3}{2}x^4 - \frac{4}{5}x^3 - \frac{3}{5}x^2 - \frac{6}{5}x - \frac{9}{10}$

Exercise 3A

1 **a** $2x^2 - 3x = 0$

$x(2x - 3) = 0$

$x = 0$ or $\frac{3}{2}$

b $3x^2 - 75 = 0$

$x^2 = 25$

$x = \pm 5$

c $5x^2 - 4x = 0$

$x(5x - 4) = 0$

$x = 0$ or $\frac{4}{5}$

d $7 + 28x^2 = 0$

no real roots

e $242x^2 + 2x = 0$

$2x(121x + 1) = 0$

$x = 0$ or $-\frac{1}{121}$

f $\sqrt{2}x^2 - \sqrt{8} = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

g $\pi x^2 - 11x = 0$

$x(\pi x - 11) = 0$

$x = 0$ or $\frac{11}{\pi}$

h $\mathrm{e}x^2 - \sqrt{3} = 0$

$x^2 = \frac{\sqrt{3}}{\mathrm{e}}$

$x = \pm\sqrt{\frac{\sqrt{3}}{\mathrm{e}}}$

2 **a** $2x^2 + 5x + 2 = 0$

$x = \frac{-5 \pm \sqrt{5^2 - 4(2)(2)}}{4}$

$x = \frac{-5 \pm \sqrt{9}}{4}$

$= \frac{-5 \pm 3}{4}$

$x = -2$ or $-\frac{1}{2}$

b $3x^2 - 10x + 3 = 0$

$x = \frac{10 \pm \sqrt{100 - 36}}{6}$

$= \frac{10 \pm \sqrt{64}}{6}$

$= \frac{10 \pm 8}{6}$

$x = 3$ or $\frac{1}{3}$

c $5x^2 + 3x - 2 = 0$

$x = \frac{-3 \pm \sqrt{9 + 40}}{10}$

$x = \frac{-3 \pm \sqrt{49}}{10}$

$x = \frac{-3 \pm 7}{10}$

$x = -1$ or $\frac{2}{5}$

d $21x^2 + 5x - 6 = 0$

$x = \frac{-5 \pm \sqrt{25 + 504}}{42}$

$x = \frac{-5 \pm \sqrt{529}}{42}$

$x = \frac{-5 \pm 23}{42}$

$x = -\frac{2}{3}$ or $\frac{3}{7}$

e $9x^2 - 6x + 35 = 0$

$x = \frac{6 \pm \sqrt{36 - 1260}}{18}$

no real roots

f $122x = 143x^2 + 24$

$143x^2 - 122x + 24 = 0$

$x = \frac{122 \pm \sqrt{14884 - 13728}}{286}$

$= \frac{122 \pm \sqrt{1156}}{286}$

$= \frac{122 \pm 34}{286}$

$x = \frac{4}{13}$ or $\frac{6}{11}$

3 a $x^2 + 4x + 2 = 0$

$x = \frac{-4 \pm \sqrt{16-8}}{2}$

$x = \frac{-4 \pm 2\sqrt{2}}{2}$

$x = -2 \pm \sqrt{2}$

b $5x^2 - 6x - 1 = 0$

$x = \frac{6 \pm \sqrt{36+20}}{10}$

$= \frac{6 \pm \sqrt{56}}{10}$

$= \frac{6 \pm 2\sqrt{14}}{10}$

$x = \frac{3 \pm \sqrt{14}}{5}$

c $3x^2 - x - 3 = 0$

$x = \frac{1 \pm \sqrt{1+36}}{6}$

$x = \frac{1 \pm \sqrt{37}}{6}$

d $2x^2 + 11x + 13 = 0$

$x = \frac{-11 \pm \sqrt{121-104}}{4}$

$x = \frac{-11 \pm \sqrt{17}}{4}$

e $11x^2 = 23x - 7$

$11x^2 - 23x + 7 = 0$

$x = \frac{23 \pm \sqrt{529-308}}{22}$

$= \frac{23 \pm \sqrt{221}}{22}$

f $29x = 5x^2 - 41$

$15x^2 - 29x - 41 = 0$

$x = \frac{29 \pm \sqrt{841+820}}{10}$

$x = \frac{29 \pm \sqrt{1661}}{10}$

4 a $x^2 + px - 2p^2 = 0$

$(x + 2p)(x - p) = 0$

$x = -2p$ or p

b $kx^2 + (k + 2)x - 2 = 0$

$(kx + 2)(x + 1) = 0$

$x = \frac{-2}{k}$ or -1

c $2ax^2 + 6 = ax + 12x$

$2ax^2 - ax - 12x + 6 = 0$

$(ax - 6)(2x - 1) = 0$

$x = \frac{6}{a}$ or $\frac{1}{2}$

d $x^2 - 2a^2 = b^2 - ax - 3ab$

$x^2 + ax + (3ab - 2a^2 - b^2) = 0$

$x = \frac{-a \pm \sqrt{a^2 - 12ab + 8a^2 + 4b^2}}{2}$

$x = \frac{-a \pm \sqrt{9a^2 - 12ab + 4b^2}}{2}$

$x = \frac{-a \pm (3a - 2b)}{2}$

$x = -2a + b$ or $a - b$

Exercise 3B

1 a $x^2 - 2x - 3 = 0$

$\Delta = 4 - 4(1)(-3)$

$\Delta = 16 > 0$

$\therefore$ 2 real roots

b $x^2 + 10x + 25 = 0$

$\Delta = 100 - 4(1)(25)$

$\Delta = 0$

$\therefore$ one real root

c $4x^2 - 3x + 2 = 0$

$\Delta = 9 - 4(4)(2)$

$\Delta = -23 < 0$

$\therefore$ no real roots

d $5x^2 - 11x + 6 = 0$

$\Delta = 121 - 4(5)(6)$

$\Delta = 1 > 0$

$\therefore$ 2 real roots

e $\frac{3}{5}x^2 - \frac{4}{7}x + \frac{2}{3} = 0$

$\Delta = \frac{16}{49} - 4\left(\frac{3}{5}\right)\left(\frac{2}{3}\right)$

$\Delta = -\frac{312}{245} < 0$

$\therefore$ no real roots

f $2x^2 + 2\sqrt{26}x + 13 = 0$

$\Delta = 104 - 4(2)(13)$

$\Delta = 0$

$\therefore$ one real root

2 **a** $x^2 - 2x - k = 0$

$\Delta = 4 + 4k$

$4 + 4k = 0$

$k = -1$

b $kx^2 + 3x - 2 = 0$

$\Delta = 9 + 8k$

$9 + 8k > 0$

$k > -\frac{9}{8}$

c $3x^2 + 5x + 2k - 1 = 0$

$\Delta = 25 - 12\,(2k - 1)$

$= 37 - 24k$

$37 - 24k < 0$

$37 < 24k$

$k > \frac{37}{24}$

d $x^2 - (3k + 2)x + k^2 = 0$

$\Delta = (3k + 2)^2 - 4k^2$

$= 5k^2 + 12k + 4$

$5k^2 + 12k + 4 = 0$

$(5k + 2)(k + 2) = 0$

$k = -\frac{2}{5} \text{ or } -2$

e $kx^2 + 2kx + k - 2 = 0$

$\Delta = 4k^2 - 4k\,(k - 2) = 8k$

$8k > 0$

$k > 0$

f $2kx^2 + (4k + 3)x + k - 3 = 0$

$\Delta = (4k + 3)^2 - 8k\,(k - 3)$

$= 16k^2 + 24k + 9 - 8k^2 + 24k$

$= 8k^2 + 48k + 9$

$8k^2 + 48k + 9 < 0$

if $8k^2 + 48k + 9 = 0$, then

$k = \frac{-48 \pm \sqrt{2304 - 288}}{16}$

$= \frac{-48 \pm 12\sqrt{14}}{16}$

$= \frac{-12 \pm 3\sqrt{14}}{4}$

$\frac{-12 - 3\sqrt{14}}{4} < k < \frac{-12 + 3\sqrt{14}}{4}$

Exercise 3C

1 **a** $x^2 - 3x + 2 = 0$

$x_1 + x_2 = 3$

$x_1\,x_2 = 2$

$\frac{2}{x_1} + \frac{2}{x_2} = \frac{2(x_2 + x_1)}{x_1\,x_2} = 3$

b $3x^2 - 5x + 1 = 0$

$x_1 + x_2 = \frac{5}{3} \quad x_1x_2 = \frac{1}{3}$

$3x_1{}^2 + 3x_2{}^2 = 3\,[(x_1 + x_2)^2 - 2x_1x_2]$

$= 3\left[\frac{25}{9} - \frac{2}{3}\right]$

$= \frac{19}{3}$

c $5x^2 + x + 3 = 0 \qquad x_1 + x_2 = \frac{-1}{5} \qquad x_1x_2 = \frac{3}{5}$

$\frac{1}{x_1^2} + \frac{1}{x_2^2} = \frac{x_2^2 + x_1^2}{x_1^2\,x_2^2} = \frac{\left(x_1 + x_2\right)^2 - 2x_1x_2}{\left(x_1x_2\right)^2}$

$= \frac{\frac{1}{25} - \frac{6}{5}}{\frac{9}{25}} = -\frac{29}{9}$

d $x^2 - 2x + 4 = 0 \qquad x_1 + x_2 = 2 \qquad x_1x_2 = 4$

$(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$

$= 4 - 16 = -12$

e $2x^2 - 4x + 3 = 0 \qquad x_1 + x_2 = 2 \qquad x_1x_2 = \frac{3}{2}$

$x^3{}_1 + x^3{}_2 = (x_1 + x_2)^3 - 3x_1x_2\,(x_1 + x_2)$

$= 8 - \frac{9}{2}(2) = -1$

f $x^2 + 3x + 1 = 0 \qquad x_1 + x_2 = -3 \qquad x_1x_2 = 1$

$\frac{1}{x_1^4} + \frac{1}{x_2^4} = \frac{x_2^4 + x_1^4}{x_1^4\,x_2^4}$

$(x_1 + x_2)^4 = x_1{}^4 + 4x_1{}^3x_2 + 6x_1{}^2x_2{}^2 + 4x_1x_2{}^3 + x_2{}^4$

$x_1{}^4 + x_2{}^4 = (x_1 + x_2)^4 - 6x_1{}^2x_2{}^2 - 4x_1x_2\,(x_1{}^2 + x_2{}^2)$

$= (x_1 + x_2)^4 - 6x_1{}^2\,x_2{}^2 - 4x_1x_2\,[(x_1 + x_2)^2 - 2x_1x_2]$

$= 81 - 6 - 4\,(9 - 2)$

$= 47$

$\therefore\ \frac{1}{x_1^4} + \frac{1}{x_2^4} = \frac{47}{1^4} = 47$

g $4x^2 - 7x + 1 = 0 \quad x_1 + x_2 = \frac{7}{4} \quad x_1x_2 = \frac{1}{4}$

$x_1^3\,x_2^2 + x_1^2x_2^3 = x_1^2x_2^2\left(x_1 + x_2\right)$

$= \frac{1}{16}\left(\frac{7}{4}\right)$

$= \frac{7}{64}$

h $7x^2 + 4x - 5 = 0 \quad x_1 + x_2 = \frac{-4}{7} \quad x_1x_2 = \frac{-5}{7}$

$(x_1 - x_2)^4 = x_1^4 - 4x_1^3x_2 + 6x_1^2x_2^2 - 4x_1x_2^3 + x_2^4$

$(x_1 + x_2)^4 = x_1^4 + 4x_1^3x_2 + 6x_1^2x_2^2 + 4x_1x_2^3 + x_2^4$

$\therefore\ (x_1 - x_2)^4 = (x_1 + x_2)^4 - 8x_1^3\,x_2 - 8x_1\,x_2^3$

$= (x_1 + x_2)^4 - 8x_1x_2\left(x_1^2 + x_2^2\right)$

$= (x_1 + x_2)^4 - 8x_1x_2\,[(x_1 + x_2)^2 - 2x_1x_2]$

$= \left(\frac{-4}{7}\right)^4 + \frac{40}{7}\left(\frac{16}{49} + \frac{10}{7}\right)$

$= \frac{256}{2401} + \frac{40}{7}\left(\frac{86}{49}\right) = \frac{24336}{2401}$

Exercise 3D

1 **a** $z = 3i \qquad \text{Re}\,(z) = 0 \qquad \text{Im}(z) = 3$

b $z = -7 \qquad \text{Re}\,(z) = -7 \qquad \text{Im}(z) = 0$

c $z = \frac{18-12i}{8} \qquad \text{Re}(z) = \frac{18}{8} = \frac{9}{4} \qquad \text{Im}(z) = -\frac{12}{8} = -\frac{3}{2}$

d $z = \frac{11}{4} + i\frac{\sqrt{7}}{5} \qquad \text{Re}(z) = \frac{11}{4} \qquad \text{Im}(z) = \frac{\sqrt{7}}{5}$

e $z = \frac{4i-2}{3\pi^2} \qquad \text{Re}(z) = \frac{-2}{3\pi^2} \qquad \text{Im}(z) = \frac{4}{3\pi^2}$

2 **a** $|12+5i| = \sqrt{144+25} = \sqrt{169} = 13$

b $|-24-7i| = \sqrt{576+49} = \sqrt{625} = 25$

c $\left|2\sqrt{2}+i\sqrt{5}\right| = \sqrt{8+5} = \sqrt{13}$

d $\left|\frac{-21+20i}{29}\right| = \sqrt{\frac{441}{841}+\frac{400}{841}} = 1$

e $\left|\frac{-3+4i}{\pi}\right| = \sqrt{\frac{9}{\pi^2}+\frac{16}{\pi^2}} = \frac{5}{\pi}$

Exercise 3E

1 $z_1 = 2+3i, \quad z_2 = \frac{3}{2} - 4i, \quad z_3 = 1-5i, \quad z_4 = \frac{3+4i}{5}$

a $z_1 + z_3 = 3 - 2i$

b $z_1 - 2z_3 = 2 + 3i - 2\left(\frac{3}{2} - 4i\right) = -1 + 11i$

c $z_2 + z_4 = \frac{21}{10} - \frac{16}{5}i$

d $5z_4 - 2z_2 = 5\left(\frac{3+4i}{5}\right) - 2\left(\frac{3}{2} - 4i\right) = 12i$

e $3z_1 + 4z_2 - z_3 - 5z_4$

$= 6 + 9i + 6 - 16i - 1 + 5i - 3 - 4i$

$= 8 - 6i$

f $z_1{\cdot}z_2 - z_3{\cdot}z_4 = (2+3i)\left(\frac{3}{5} - 4i\right) - (1-5i)\left(\frac{3+4i}{5}\right)$

$= 3 - 8i + \frac{9}{2}i - 12i^2 - \frac{1}{5}(3+4i-15i-20i^2)$

$= 15 - \frac{7}{2}i - \frac{23}{5} + \frac{11}{5}i$

$= \frac{52}{5} - \frac{13}{10}i$

g $z_3^2 - \frac{2}{3}(z_2 \cdot z_4) = (1-5i)^2 - \frac{2}{3}\left(\frac{3}{2} - 4i\right)\left(\frac{3+4i}{5}\right)$

$= 1 - 10i + 25i^2 - \frac{2}{15}\left(\frac{9}{2} + 6i - 12i - 16i^2\right)$

$= -24 - 10i - \frac{2}{15}\left(\frac{41}{2} - 6i\right)$

$= -\frac{401}{15} - \frac{46i}{5}$

Exercise 3F

1 $z_1 = 1 + 4i, z_2 = 2 - i, \; z_3 = \frac{1}{2} - \frac{5}{2}i \;\; z_4 = \frac{2i-1}{3}$

a $\frac{z_1}{z_2} = \frac{1+4i}{2-i} \times \frac{2+i}{2+i} = \frac{2+i+8i-4}{4+1} = \frac{-2}{5} + \frac{9}{5}i$

b $\frac{z_1^*}{z_2} = \frac{1-4i}{1+4i} \times \frac{1-4i}{1-4i} = \frac{1-4i-4i-16}{1+16} = \frac{-15}{17} - \frac{-8}{17}i$

c $\frac{z_2.z_4}{z_3} = \frac{(2-i)\frac{(2i-1)}{3}}{\frac{1}{2}-\frac{5}{2}i} = \frac{2}{3}\frac{(4i-2+2+i)}{1-5i}$

$= \frac{2}{3} \times \frac{5i}{(1-5i)} \times \frac{1+5i}{1+5i} = \frac{2}{3}\left(\frac{5i-25}{1+25}\right) = \frac{-50+10i}{78}$

$= \frac{-25}{39} + \frac{5}{39}i$

d $\frac{3z_1 - 2z_3}{z_2 + 3z_4} = \frac{3+12i-1+5i}{2-i+2i-1} = \frac{2+17i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{2-2i+17i+17}{1+1}$

$= \frac{19}{2} + \frac{15}{2}i$

e $\frac{z_1^2}{\left(z_2^*\right)^2} = \frac{(1+4i)^2}{(2+i)^2} = \frac{1+8i-16}{4+4i-1} = \frac{-15+8i}{3+4i} \times \frac{3-4i}{3-4i}$

$= \frac{-45+60i+24i+32}{9+16}$

$= \frac{-13}{25} + \frac{84}{25}i$

2 **a** $(2+i)\,(a+ib) = 11 - 2i$

$a + ib = \frac{11-2i}{2+i} \times \frac{2-i}{2-i}$

$= \frac{22-11i-4i-2}{4+1}$

$= 4 - 3i \quad a = 4, b = -3$

b $\frac{a+ib}{2-5i} = -3 + 2i$

$a + ib = (-3+2i)\,(2-5i)$

$= -6 + 15i + 4i + 10$

$= 4 + 19i$

$a = 4, b = 19$

c $(3i-2)\,(a+ib) = 3 + 28i$

$a + ib = \frac{3+28i}{-2+3i} \times \frac{-2-3i}{-2-3i} = \frac{-6-9i-56i+84}{4+9}$

$= 6 - 5i$

$a = 6, b = -5$

d $\left(\frac{1}{2} + \frac{3}{4}i\right)(a+ib) = -3 + 2i$

$a + ib = \frac{-12+8i}{2+3i} \times \frac{2-3i}{2-3i} = \frac{-24+36i+16i+24}{4+9}$

$= 4i$

$a = 0, b = 4$

3 **a** $\frac{3-2i}{4}$ $\quad \text{Re}(z)=\frac{3}{4} \quad \text{Im}(z)=-\frac{1}{2}$

b $\frac{5i-2}{3i}\times\frac{-3i}{-3i}=\frac{15+6i}{9} \quad \text{Re}(z)=\frac{5}{3} \quad \text{Im}(z)=\frac{2}{3}$

c $\frac{1}{3i}+\frac{2}{1+i}=\frac{1+i+6i}{3i(1+i)}=\frac{1+7i}{-3+3i}\times\frac{-3-3i}{-3-3i}$

$=\frac{-3-3i-21i+21}{9+9}=1-\frac{4}{3}i$

$\text{Re}(z)=1 \quad \text{Im}(z)=\frac{-4}{3}i$

d $\frac{2-3i}{2+3i}-\frac{2+3i}{2-3i}=\frac{(2-3i)^2-(2+3i)^2}{4+9}$

$=\frac{4-12i-9-4-12i+9}{13}=\frac{-24i}{13}$

$\text{Re}(z)=0 \quad \text{Im}(z)=\frac{-24}{13}$

4 $z_1 = 1 + 3i \qquad z_2 = 3 - i$

a $z_1 \cdot z_2 + z_1 \cdot z_2^* = z_1(z_2 + z_2^*)$

$= (1 + 3i)(3 - i + 3 + i)$

$= 6 + 18i$

b $z_1 \cdot z_2 - z_1^* \cdot z_2 = (z_1 - z_1^*) \cdot z_2$

$= (1 + 3i - (1 - 3i))\,(3 - i)$

$= 6i\,(3 - i)$

$= 6 + 18i$

c $z_1 \cdot z_2 + (z_1 \cdot z_2)^* = (1 + 3i)(3 - i) + [(1 + 3i)(3 - i)]^*$

$= (6 + 8i) + (6 - 8i)$

$= 12$

5 **a** $(z + 1)i = (z + 2i)(3 + 2i)$

$zi + i = 3z + 2zi + 6i - 4$

$i - 6i + 4 = 3z + zi$

$z=\frac{4-5i}{3+i}\times\frac{3-i}{3-i}=\frac{12-4i-15i-5}{9+1}$

$=\frac{7}{10}-\frac{19}{10}i$

b $(2z - 1)(1 + i) = (z - 1)(2 + 3i)$

$2z + 2zi - 1 - i = 2z + 3zi - 2 - 3i$

$\therefore\ 1 + 2i = zi$

$z=\frac{1+2i}{i}\times\frac{-i}{i}$

$z = 2 - i$

c $\frac{z-3i+2}{4+3i}=\frac{z-1}{1+i}$

$(z - 3i + 2)(1 + i) = (z - 1)(4 + 3i)$

$z + zi - 3i + 3 + 2 + 2i = 4z + 3zi - 4 - 3i$

$z + zi - i + 5 = 4z + 3zi - 4 - 3i$

$9 + 2i = 3z + 2zi$

$z=\frac{9+2i}{3+2i}\times\frac{3-2i}{3-2i}=\frac{27-18i+6i+4}{9+4}$

$z=\frac{31}{13}-\frac{12}{13}i$

d $\frac{38-2i}{2+i}=\frac{28+5}{10+15i}$

$(3z - 2i)(10 + 15i) = (2z + 5)(2 + i)$

$30z + 45zi - 20i + 30 = 4z + 2zi + 10 + 5i$

$26z + 43zi = -20 + 25i$

$z=\frac{-20+25i}{26+43i}\times\frac{26-43i}{26-43i}=\frac{-520+860i+650i+1075}{26^2+43^2}$

$=\frac{555+1510i}{2525}=\frac{111}{505}+\frac{302}{505}i$

6 $\frac{z}{2-7i}\in\mathbb{R}.\quad$ Let $z = a + bi$

$\frac{a+bi}{2-7i}\times\frac{2+7i}{2+7i}=\frac{2a+7ai+2bi-7b}{4+49}$

$\text{Im}\left(\frac{z}{2-7i}\right)=0 \quad \therefore\ 7a + 2b = 0$

$\therefore\ 7\text{Re}(z) + 2\,\text{Im}(z) = 0$

7 Let $z = a + bi, \quad z^* = a - bi$

$\frac{3-5i}{z^*}=\frac{3-5i}{a-bi}\times\frac{a+bi}{a+bi}=\frac{3a+3bi-5ai+5b}{a^2+b^2}$

$\text{Re}\left(\frac{3-5i}{z^*}\right)=0 \quad \therefore\ 3a + 2b = 0$

$\therefore\ 3\text{Re}(z) + 5\,\text{Im}(z) = 0$

8 **a** $|\,z\,| - z = 4 + 3i.\quad$ Let $z = a + ib$

$\sqrt{a^2+b^2} = 4 + 3i + a + ib$

$= (4 + a) + i\,(3 + b)$

equating real and imaginary parts,

$\sqrt{a^2+b^2} = 4 + a$

$3 + b = 0 \quad \therefore\ b = -3$

$a^2 + 9 = (4 + a)^2$

$a^2 + 9 = 16 + 8a + a^2$

$8a = -7$

$a=-\frac{7}{8} \qquad \therefore \quad z=-\frac{7}{8}-3i$

b $|\,z\,| + iz = 2 - i \quad$ Let $z = a + ib$

$\sqrt{a^2+b^2} = 2 - i - i(a + ib)$

$= 2 - i - ia + b$

$= (2 + b) - (1 + a)i$

$\sqrt{a^2+b^2} = 2 + b \qquad 1 + a = 0$

$a = -1$

$1 + b^2 = (2 + b)^2$

$1 + b^2 = 4 + 4b + b^2 \qquad 4b = -3$

$b=\frac{-3}{4} \qquad \therefore \quad z=-1-\frac{3}{4}i$

c $z^2 - z^* = 0.\quad$ Let $z = a + ib$

$(a + ib)^2 - (a - ib) = 0$

$a^2 + 2abi - b^2 - a + ib = 0$

$a^2 - b^2 - a = 0 \qquad 2ab + b = 0$

$b(2a + 1) = 0$

$b = 0$ or $a=-\frac{1}{2}$

If $b = 0$, $a^2 - a = 0$, $a(a - 1) = 0$, $a = 0$ or 1

If $a = -\frac{1}{2}$, $\frac{1}{4} - b^2 + \frac{1}{2} = 0$, $b^2 = \frac{3}{4}$, $b = \pm\frac{\sqrt{3}}{2}$

$z = 0, z = 1, z = \frac{-1}{2} + \frac{\sqrt{3}}{2}i, z = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$

9 Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$

a $z_1 \cdot z_2 = (a_1a_2 - b_1b_2) + i\,(a_1b_2 + a_2b_1)$

$|z_1 \cdot z_2| =$

$$\sqrt{(a_1a_2)^2 - 2a_1a_2b_1b_2 + (b_1b_2)^2 + (a_1b_2)^2 + 2a_1b_2a_2b_1 + (a_2b_1)^2}$$

$$= \sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2}$$

$$|z_1| \cdot |z_2| = \sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}$$

$$= \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$$

$$= \sqrt{a_1^2a_2^2 + a_1^2b_2^2 + b_1^2a_2^2 + b_1^2b_2^2}$$

$\therefore\ |z_1 . z_2| = |z_1| . |z_2|$

b $$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{a_2 - ib_2}{a_2 - ib_2} = \frac{(a_1a_2 + b_1b_2) + i(a_2b_1 - a_1b_2)}{a_2^2 + b_2^2}$$

$$\left|\frac{z_1}{z_2}\right| = \sqrt{\frac{(a_1a_2 + b_1b_2)^2}{(a_2^2 + b_2^2)^2} + \frac{(a_2b_1 - a_1b_2)^2}{(a_2^2 + b_2^2)^2}}$$

$$= \frac{\sqrt{a_1^2a_2^2 + 2a_1a_2b_1b_2 + b_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2 + a_1^2b_2^2}}{a_2^2 + b_2^2}$$

$$= \frac{\sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_2^2b_1^2 + a_1^2b_2^2}}{a_2^2 + b_2^2}$$

$$= \frac{\sqrt{\left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right)}}{a_2^2 + b_2^2} = \frac{\sqrt{a_1^2 + b_1^2}}{\sqrt{a_2^2 + b_2^2}} = \frac{|z_1|}{|z_2|}$$

$$\therefore \frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|}$$

c Let $p(n)$ be $|z^n| = |z|^n$

Step 1: When $n = 1$, $|z^n| = |z| = |z|^1$ so $p(1)$ is true.

Step 2: Assume $p(n)$ is true when $n = k$, i.e. $|z^k| = |z|^k$.

Step 3: Prove $p(n)$ is true when $n = k + 1$.

Proof : $|z^{k+1}| = |z^k \cdot z|$

$= |z^k|\,|z|$

$= |z|^k\,|z|$

$= |z|^{k+1}$

Since $p(1)$ is true and if $p(k)$ is true then $p(k + 1)$ is true, by mathematical induction $p(n)$ is true.

d Let $z_1 = a + bi$ and $z_2 = c + di$.

Then $z_1 + z_2 = (a + c) + (b + d)i$.

From the properties of a triangle, $|z_1 + z_2|$

$\le \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2}$

$\therefore |z_1 + z_2| \le |z_1| + |z_2|$.

10 Let $z = a + bi$, $z_1 = a_1 + b_1i$, $z_2 = a_2 + b_2i$

a $z^* = a - bi \quad (z^*)^* = a + bi = z$

$\therefore (z^*)^* = z$

b $z_1 + z_2 = (a_1 + a_2) + i\,(b_1 + b_2)$

$(z_1 + z_2)^* = (a_1 + a_2) - i\,(b_1 + b_2)$

$= (a_1 - ib_1) + (a_2 - ib_2) = z_1^* + z_2^*$

$\therefore (z_1 + z_2)^* = z_1^* + z_2^*$

c $z_1 . z_2 = (a_1a_2 - b_1b_2) + i\,(a_1b_2 + a_2b_1)$

$(z_1 . z_2)^* = (a_1a_2 - b_1b_2) - i\,(a_1b_2 + a_2b_1)$

$z_1^* . z_2^* = (a_1 - ib_1)(a_2 - ib_2)$

$= (a_1a_2 - b_1b_2) - i\,(a_1b_2 + a_2b_1)$

$\therefore (z_1 . z_2)^* = z_1^* . z_2^*$

d $z . z^* = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$

$\therefore z . z^* = |z|^2$

e $|z^*| = |a - ib| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$

$\therefore |z^*| = |z|$

Exercise 3G

1 **a** $i^5 + i^8 + i^{14} + i^{19} = i + 1 - 1 - i = 0$

b $i^{123} + i^{172} + i^{256} + i^{375} = -i + 1 + 1 - i = 2 - 2i$

c $(2 - i^{53})\,(3 + 2i^{89}) = (2 - 1)(3 + 2i)$

$= 6 + 4i - 3i + 2$

$= 8 + i$

d $$\frac{4i^{2010} - 3i^{2011}}{2i^{2012} + 5i^{2013}} = \frac{-4 + 3i}{2 + 5i}$$

$$= \frac{-4 + 3i}{2 + 5i} \times \frac{2 - 5i}{2 - 5i} = \frac{-8 + 20i + 6i + 15}{4 + 25}$$

$$= \frac{7}{29} + \frac{26i}{29}$$

e $$\frac{i + i^2 + i^3 + ... + i^{2011}}{i . i^2 . i^3 .... i^{2011}} = \frac{\frac{i(1 - i^{2011})}{1 - i}}{i^{(1+2+3+....+2011)}}$$

$$= \frac{i(1 + i)}{(1 - i)i^{\frac{2011}{2}(2012)}} = \frac{i(1 + i)}{(1 - i)i^{2023066}}$$

$$= \frac{i(1 + i)}{(1 - i)(-1)} = \frac{-1 + i}{-1 + i} = 1$$

f $$\frac{i^2 + i^4 + i^6 + ... + i^{2010}}{i^2 . i^4 . i^6 .... i^{2010}} = \frac{-1 + 1 - 1 + 1 ...... - 1}{i^{(2+4+6+....+2010)}}$$

$$= \frac{-1}{i^{\frac{1005}{2}(2012)}} = \frac{-1}{i^{1011030}} = \frac{-1}{-1} = 1$$

2 **a** $(2 + 3i)^2 + (1 - 4i)^2 = 4 + 12i - 9 + 1 - 8i - 16$

$= -20 + 4i$

b $(3 + 2i)^2 + (3 - 2i)^2 = 9 + 12i - 4 + 9 - 12i - 4$

$= 10$

c $(3 + 2i)^3 = 3^3 + 3(3)^2(2i) + 3(3)(2i)^2 + (2i)^3$

$= 27 + 54i - 36 - 8i$

$= -9 + 46i$

$(3 - 2i)^3 = 3^3 + 3(3)^2(-2i) + 3(3)(-2i)^2 + (-2i)^3$

$= 27 - 54i - 36 + 8i$

$= -9 - 46i$

$\therefore (3 + 2i)^3 + (3 - 2i)^3 = -18$

d $(1 + i)^4 = (1 + i)^2(1 + i)^2 = (1 + 2i - 1)$

$(1 + 2i - 1) = -4$

$(1 - i)^4 = (1 - i)^2(1 - i)^2 = (1 - 2i - 1)$

$(1 - 2i - 1) = -4$

$\therefore (1 + i)^4 + (1 - i)^4 = -8$

3 **a** $\sqrt{3+4i} = x + iy$

$3 + 4i = x^2 - y^2 + 2ixy$

$x^2 - y^2 = 3 \quad 2xy = 4$

$y = \frac{2}{x}$

$x^2 - \frac{4}{x^2} = 3 \quad \therefore x^4 - 3x^2 - 4 = 0$

$(x^2 - 4)(x^2 + 1) = 0$

$\therefore x = \pm 2$

$\sqrt{3+4i} = 2 + i$ or $-2 - i$

b $\sqrt{12i-5} = x + iy$

$12i - 5 = x^2 - y^2 + 2ixy$

$x^2 - y^2 = -5 \quad 2xy = 12$

$y = \frac{6}{x}$

$x^2 - \frac{36}{x^2} = -5 \quad \therefore x^4 + 5x^2 - 36 = 0$

$(x^2 + 9)(x^2 - 4) = 0$

$x = \pm 2$

$\sqrt{12i-5} = 2 + 3i$ or $-2 - 3i$

c $\sqrt{\frac{5}{4}+3i} = x + iy$

$\frac{5}{4} + 3i = x^2 - y^2 + 2ixy$

$x^2 - y^2 = \frac{5}{4}, 2xy = 3$

$\therefore y = \frac{3}{2x}$

$x^2 - \frac{9}{4x^2} = \frac{5}{4}$

$4x^4 - 5x^2 - 9 = 0$

$(4x^2 - 9)(x^2 + 1) = 0$

$x = \pm\frac{3}{2}$

$\sqrt{\frac{5}{4}+3i} = \frac{3}{2} + i$ or $-\frac{3}{2} - i$

d $\sqrt{\frac{55}{144} - \frac{1}{3}i} = x + iy$

$\frac{55}{144} - \frac{1}{3}i = x^2 - y^2 + 2ixy$

$x^2 - y^2 = \frac{55}{144} \quad 2xy = \frac{-1}{3}$

$y = \frac{-1}{6x}$

$x^2 - \frac{1}{36x^2} = \frac{55}{144}$

$36x^4 - \frac{55}{4}x^2 - 1 = 0 \quad \therefore 144x^4 - 55x^2 - 4 = 0$

$(9x^2 - 4)(16x^2 + 1) = 0$

$x = \pm\frac{2}{3}$

$x = \frac{2}{3}, \ y = \frac{-1}{4} \qquad x = \frac{-2}{3}, \ y = \frac{-1}{4}$

$\sqrt{\frac{55}{144} - \frac{1}{3}i} = \frac{-2}{3} + \frac{1}{4}i$ or $\frac{2}{3} - \frac{1}{4}i$

e $\sqrt{i} = x + iy$

$i = x^2 - y^2 + 2ixy$

$x^2 = y^2 \quad 2xy = 1$

$y = \frac{1}{2x}$

$x^2 = \frac{1}{4x^2}, x^4 = \frac{1}{4} \quad x = \pm\frac{1}{\sqrt{2}} \quad y = \pm\frac{1}{\left(\frac{2}{\sqrt{2}}\right)} = \pm\frac{1}{\sqrt{2}}$

$\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ or $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

f $\sqrt{-i} = x + iy$

$-i = x^2 - y^2 + 2ixy$

$x^2 = y^2 \quad 2xy = -1$

$y = \frac{-1}{2x}$

As in **e**, $x = \pm\frac{1}{\sqrt{2}}, \sqrt{-i} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ or $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

4 **a** $(1 + i)^{2n} = [(1 + i)^2]^n = [1 + 2i - 1]^n$

$\therefore (1 + i)^{2n} = (2i)^n$

b $(1 + i)^{2n+1} = (1 + i)(1 + i)^{2n} = (1 + i)(2i)^n$

from **a** $\therefore (1 + i)^{2n+1} = (1 + i)(2i)^n$

Exercise 3H

1 $f(x) = 2x^2 + 3x + 1 \quad g(x) = 3x^2 - 2x - 5$

a $\lambda \cdot f(x) + \mu \cdot g(x) = 2\lambda x^2 + 3\lambda x + \lambda + 3\mu x^2 - 2\mu x - 5\mu$

$= (2\lambda + 3\mu)x^2 + (3\lambda - 2\mu)x + (\lambda - 5\mu)$

$2\lambda + 3\mu = 0$

$3\lambda - 2\mu = 13$

$\lambda - 5\mu = 13$

$\lambda = 3, \mu = -2$

b $2\lambda + 3\mu = 26$

$3\lambda - 2\mu = 26$

$\lambda - 5\mu = 0$

$\lambda = 10, \mu = 2$

2 **a**

	x^3	$-2x$
x^2	x^5	$-2x^3$
2	$2x^3$	$-4x$

$f(x) \cdot g(x) = x^5 - 4x$

b

	$27x^3$	$-36x^2$	$48x$	-64
$3x^2$	$81x^5$	$-108x^4$	$144x^3$	$-192x^2$
$7x$	$189x^4$	$-252x^3$	$336x^2$	$-448x$
4	$108x^3$	$-144x^2$	$192x$	-256

$f(x)\cdot g(x) = 81x^5 + 81x^4 - 256x - 256$

3 $f(x)\cdot g(x) = (ax^2 - 3x + 5)(7x^2 + bx - 3)$

$= 7ax^4 + abx^3 - 3ax^2 - 21x^3 - 3bx^2 + 9x$
$+ 35x^2 + 5bx - 15$

$= 7ax^4 + (ab - 21)x^3 + (-3a - 3b + 35)x^2$
$+ (9 + 5b)x - 15$

$7a = 14$

$ab - 21 = -17$

$-3a - 3b + 35 = 23$

$9 + 5b = 19 \quad a = 2 \quad b = 2$

4 $f(x)\cdot g(x) = (x^3 + ax^2 - x + 2)(2x^2 + bx + c)$

$= 2x^5 + bx^4 + cx^3 + 2ax^4 + abx^3$
$+ acx^2 - 2x^3 - bx^2 - cx + 4x^2 + 2bx + 2c$

$2a + b = -5$

$c + ab - 2 = 3$

$ac - b + 4 = 5$

$-c + 2b = -8$

$a = -1 \quad b = -3 \quad c = 2$

5 $(x^2 + px + q)^2 = (x^2 + px + q)(x^2 + px + q)$

$= x^4 + px^3 + qx^2 + px^3 + p^2x^2 + pqx + qx^2$
$+ pxq + q^2$

$= x^4 + 2px^3 + (2q + p^2)x^2 + 2pqx + q^2$

$2p = 6 \qquad p = 3$

$2q + p^2 = a \qquad q = \pm 2$

$2pq = b \qquad$ If $q = 2, \quad a = 13, \quad b = 12$

$q^2 = 4 \qquad$ If $q = -2, \quad a = 5, b = -12$

$a = 13, b = 12, f(x) = (x^2 + 3x + 2)^2$

or $a = 5, b = -12, f(x) = (x^2 + 3x - 2)^2$

6 $f(x) = x^3 + 12x^2 + 6x + 3$

$g(x) = f(x - 2)$

$= (x - 2)^3 + 12(x - 2)^2 + 6(x - 2) + 3$

$= x^3 + 3x^2(-2) + 3x(-2)^2 + (-2)^3 +$
$12(x^2 - 4x + 4) + 6x - 12 + 3$

$= x^3 - 6x^2 + 12x - 8 + 12x^2 - 48x + 48 + 6x - 9$

$g(x) = x^3 + 6x^2 - 30x + 31$

7 $f(2x - 1) = 16x^4 - 32x^3 + 12x^2$

Let $y = 2x - 1,\ x = \frac{y+1}{2}$

$$f(y) = 16\frac{(y+1)^4}{16} - 32\frac{(y+1)^3}{8} + 12\frac{(y+1)^2}{4}$$

$= y^4 + 4y^3 + 6y^2 + 4y + 1 - 4(y^3 + 3y^2 + 3y + 1)$
$+ 3(y^2 + 2y + 1)$

$f(y) = y^4 - 3y^2 - 2y$

$\therefore f(x) = x^4 - 3x^2 - 2x$

8 $f(0) = 4 \Rightarrow e = 4$

$f(10) = 32584 \Rightarrow 10000a + 1000b + 100c + 10d + 4$
$= 32584$

Since $a, b, c, d, \in \mathbb{Z}^+$ and are less than 10,

$a = 3, b = 2, c = 5, d = 8$

$\Rightarrow f(x) = 3x^4 + 2x^3 + 5x^2 + 8x + 4$

Exercise 3I

1 **a**

$$\begin{array}{r} x^3+3x^2+2x-1 \\ x+2\overline{\big)x^4+5x^3+8x^2+3x-2} \\ x^4+2x^3 \qquad\qquad\qquad \\ 3x^3+8x^2 \qquad\qquad \\ 3x^3+6x^2 \qquad\qquad \\ 2x^2+3x \qquad \\ 2x^2+4x \qquad \\ -x-2 \\ -x-2 \end{array}$$

$q(x) = x^3 + 3x^2 + 2x - 1$

b

$$\begin{array}{r} x^3+3x^2+2x-1 \\ x^2+0x-1\overline{\big)x^5+3x^4+x^3-4x^2-2x+1} \\ x^5+0x^4-x^3 \qquad\qquad\qquad \\ 3x^4+2x^3-4x^2 \qquad\qquad \\ 3x^4+0x^3-3x^2 \qquad\qquad \\ 2x^3-x^2-2x \qquad \\ 2x^3+0x^2-2x \qquad \\ -x^2+0x+1 \\ -x^2+0x+1 \end{array}$$

$q(x) = x^3 + 3x^2 + 2x - 1$

c

$$\begin{array}{r} 2x^3-5x^2+4x-1 \\ x^2+x+1\overline{\big)2x^5-3x^4+x^3-2x^2+3x-1} \\ 2x^5+2x^4+2x^3 \qquad\qquad\qquad \\ -5x^4-x^3-2x^2 \qquad\qquad \\ -5x^4-5x^3-5x^2 \qquad\qquad \\ -4x^3+3x^2+3x \qquad \\ 4x^3+4x^2+4x \qquad \\ -x^2-x-1 \\ -x^2-x-1 \end{array}$$

$q(x) = 2x^3 - 5x^2 + 4x - 1$

2 **a**

$$x+1\overline{\smash{)}2x^4+5x^3+4x^2+4x+3}$$ with quotient $2x^3+3x^2+x+3$

$2x^4+2x^3$

$3x^3+4x^2$

$3x^3+3x^2$

x^2+4x

x^2+x

$3x+3$

$3x+3$

$q(x) = 2x^3 + 3x^2 + x + 3 \quad r(x) = 0$

b

$$x^2+2x+3\overline{\smash{)}3x^4+4x^3+6x^2-2x+6}$$ with quotient $3x^2-2x+1$

$3x^4+6x^3+9x^2$

$-2x^3-3x^2-2x$

$-2x^3-4x^2-6x$

x^2+4x+6

x^2+2x+3

$2x+3$

$q(x) = 3x^2 - 2x + 1 \quad r(x) = 2x + 3$

c

$$x^2+x+1\overline{\smash{)}x^6+0x^5+0x^4+0x^3+0x^2+x-1}$$ with quotient x^4-x^3+x-1

$x^6+x^5+x^4$

$-x^5-x^4+0x^3$

$-x^5-x^4-x^3$

x^3+0x^2+x

x^3+x^2+x

$-x^2+0x-1$

$-x^2-x-1$

x

$q(x) = x^4 - x^3 + x - 1 \quad r(x) = x$

Exercise 3J

1 **a** $x^3 - x^2 - 4x - 5 = (x^2 + 2x + 2)(x - 3) + 1$,

$q(x) = x^2 + 2x + 2 \qquad r(x) = 1$

b $2x^3 + 5x^2 + 4x + 3 = (2x^2 + 3x + 1)(x + 1) + 2$,

$q(x) = 2x^2 + 3x + 1 \qquad r(x) = 2$

c $x^5 - 3x^3 - 2x + 1 = (x^4 - 2x^3 + x^2 - 2x + 2)(x + 2) - 3$,

$q(x) = x^4 - 2x^3 + 2x^2 - 2x + 2, \qquad r(x) = -3$

d $3x^6 - 2x^4 + 5x^2 - 2 = (3x^5 + 3x^4 + x^3 + x^2 + 6x + 6)(x - 1) + 4$,

$q(x) = 3x^5 + 3x^4 + x^3 + x^2 + 6x + 6, \quad r(x) = 4$

2 $f(2) = 4 \times 16 - 27 \times 4 + 25 \times 2 - 6 = 0$

$\therefore (x - 2)$ is a factor.

$f(-3) = 4 \times 81 - 27 \times 9 - 25 \times 3 - 6 = 0$

$\therefore (x + 3)$ is a factor.

Exercise 3K

1 **a** $q(x) = x^4 - x^3 + x^2 + 2x + 1 \qquad r(x) = -2$

b $q(x) = x^3 + x^2 + x - 1 \qquad r(x) = 7$

2 $f(x) = (x^2 + 2x - 1)(3x - 4) + x + 2$

$= 3x^3 - 4x^2 + 6x^2 - 8x - 3x + 4 + x + 2$

$f(x) = 3x^3 + 2x^2 - 10x + 6$

3 $f(x) = x^5 - 4x^4 + 3x^3 + 2x^2 - 3x + a$

$f(3) = 0 \quad 243 - 324 + 81 + 18 - 9 + a = 0$

$a = -9$

4 $f(x) = x^5 - 2x^4 + 2x^3 + bx - 1$

$f(1) = 0 \quad 1 - 2 + 2 + b - 1 = 0$

$\therefore b = 0$

5 $f(x) = 4x^3 + 5x^2 + ax + b$

$f(-2) = 0 \quad f(1) = 6$

$-32 + 20 - 2a + b = 0 \quad 4 + 5 + a + b = 6$

$-2a + b = 12 \qquad a + b = -3$

$a = -5, b = 2$

6 $f(x) = (x^2 - 2x - 3)\,q(x) + ax + b$

$f(x) = (x - 3)(x + 1)\,q(x) + ax + b$

$f(3) = 2 \qquad \therefore 2 = 3a + b$

$f(-1) = -4 \qquad \therefore -4 = -a + b$

$\therefore a = 1.5 \quad b = -2.5$

$\therefore$ remainder $= \frac{3}{2}x - \frac{5}{2}$

7 $f(x) = x^{2011} + x^{2010} + \ldots + x + 1 = 0$

remainder $= f(-1) = -1 + 1 - 1 + \ldots - 1 + 1 = 0$

$\therefore$ remainder $= 0$

8 $f(x) = (x + 1)^{2n} + (x + 2)^n - 1$

$f(-1) = 0^{2n} + (1)^n - 1 = 0 + 1 - 1 = 0$

$\therefore f(x)$ is divisible by $(x + 1)$

$f(-2) = (-1)^{2n} + 0^n - 1 = 1 + 0 - 1 = 0$

$\therefore f(x)$ is divisible by $(x + 2)$

$(x + 1)(x + 2) = x^2 + 3x + 2$

$\therefore f(x)$ is divisible by $x^2 + 3x + 2$

9 $f(x) = (ax - b)\,q(x) + r(x)$

$f\left(\frac{b}{a}\right) = \left(\frac{ab}{a} - b\right)q\left(\frac{b}{a}\right)$ + remainder

$= (0)\,q\left(\frac{b}{a}\right)$ + remainder

$\therefore$ remainder $= f\left(\frac{b}{a}\right)$

Exercise 3L

1 **a** $f(x) = 2x^4 + 3x^3 - 10x^2 - 12x + 8$

$= (x + 2)^2 g(x)$

$= (x^2 + 4x + 4)(2x^2 - 5x + 2)$

$= (x + 2)(x + 2)(2x - 1)(x - 2)$

b $f(x) = 12x^3 - 32x^2 + 23x - 5$

$= (2x - 1)^2 g(x)$

$= (4x^2 - 4x + 1)(3x - 5)$

$= (2x - 1)^2(3x - 5)$

2 **a** $f(x) = x(x - 1)(x - 3)(x - 5)$

$= (x - 1)(x^2 - 8x + 15)$

$= x^3 - 9x^2 + 23x - 15$

b $f(x) = x(x + 2)(x + 1)(x - 1)$

$= x(x + 2)(x^2 - 1)$

$= x(x^3 + 2x^2 - x - 2)$

$= x^4 + 2x^3 - x^2 - 2x$

c $f(x) = (3x + 2)(x - 1)(x - 2)(x - 3)$

$= (3x^2 - x - 2)(x^2 - 5x + 6)$

$= 3x^4 - 16x^3 + 21x^2 + 4x - 12$

3 **a** $f(x) = (x^2 - 2)(x^2 - 3) = x^4 - 5x^2 + 6$

b $f(x) = (2x + 1)(4x - 3)(x^2 - 5)$

$= (2x + 1)(4x^3 - 3x^2 - 20x + 15)$

$= 8x^4 - 2x^3 - 43x^2 + 10x + 15$

c $f(x) = (5x + 3)(x - (1 - \sqrt{2}))(x - (1 + \sqrt{2}))$
$(x^3 - 3)$

$= (5x + 3)(x^2 - 2x - 1)(x^3 - 3)$

$= (5x + 3)(x^5 - 2x^4 - x^3 - 3x^2 + 6x + 3)$

$= 5x^6 - 7x^5 - 11x^4 - 18x^3 + 21x^2 + 33x + 9$

4 **a** $f(x) = x^3 - 2x^2 - 5x + 6$

$= (x - 1)(x^2 - x - 6)$

$= (x - 1)(x - 3)(x + 2)$

b $f(x) = 2x^3 - x^2 - 7x + 6$

$= (x - 1)(2x^2 + x - 6)$

$= (x - 1)(2x - 3)(x + 2)$

c $f(x) = 5x^4 - 12x^3 - 14x^2 + 12x + 9$

$= (x - 1)(x + 1)g(x)$

$= (x^2 - 1)(5x^2 - 12x - 9)$

$= (x - 1)(x + 1)(5x + 3)(x - 3)$

Exercise 3M

1 **a** $(x - 2i)(x + 2i) = x^2 + 4$

$f(x) = x^3 + 3x^2 + 4x + 12$

$= (x^2 + 4)(x + 3)$

remaining zeros are $-2i$ and -3

b $(x -(1 - 2i))(x - (1 + 2i)) = x^2 - 2x + 5$

$f(x) = x^3 - 6x^2 + 13x - 20$

$= (x^2 - 2x +5)(x - 4)$

remaining zeros are $1 + 2i$ and 4

c $\left(x-\left(-\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)\right)\left(x-\left(-\frac{3}{2}-\frac{\sqrt{3}}{2}\right)i\right) = x^2 + 3x + 3$

$f(x) = 5x^3 + 17x^2 + 21x + 6$

$= (x^2 + 3x + 3)(5x + 2)$

remaining zeros are $-\frac{3}{2}-\frac{\sqrt{3}}{2}i$ and $-\frac{2}{5}$

d $(x - i)(x + i) = x^2 + 1$

$f(x) = x^4 - 6x^3 + 5x^2 - 4x + 4$

$= (x^2 + 1)(x^2 - 4x + 4)$

$= (x + i)(x - i)(x - 2)^2$

remaining zeros are i, 2 and 2

e $(x -(-1 - 3i))(x -(-1 + 3i)) = x^2 + 2x + 10$

$f(x) = 2x^4 + 3x^3 + 17x^2 - 12x - 10$

$= (x^2 + 2x + 10)(2x^2 - x - 1)$

$= (x -(-1 -3i))(x -(-1 + 3i))(2x + 1)(x - 1)$

remaining zeros are $-1+3i, -\frac{1}{2}, 1$

f $(x -(-2 + i))(x -(-2 -i)) = x^2 + 4x + 5$

$f(x) = 2x^4 + 9x^3 + 11x^2 - 7x - 15$

$= (x^2 + 4x + 5)(2x^2 + x - 3)$

$= (x -(-2 + i))(x -(-2 - i))$
$(2x + 3)(x - 1)$

remaining zeros are $-2-i, -\frac{3}{2}, 1$

g $\left(x-\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}i\right)\right)\left(x-\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}i\right)\right) = x^2 + x + \frac{3}{2}$

$f(x) = 6x^4 + 26x^3 + 35x^2 + 36x + 9$

$= (2x^2 + 2x + 3)(3x^2 + 10x + 3)$

$= 2\left(x-\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}i\right)\right)\left(x-\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}i\right)\right)(3x+1)(x+3)$

remaining zeros are $-\frac{1}{2}-\frac{\sqrt{5}}{2}i, -\frac{1}{3}, -3$

h $\left(x-\left(\frac{1}{3}+\frac{\sqrt{2}}{3}i\right)\right)\left(x-\left(\frac{1}{3}-\frac{\sqrt{2}}{3}i\right)\right) = x^2 - \frac{2}{3}x + \frac{1}{3}$

$f(x) = 3x^4 - 2x^3 + 4x^2 - 2x + 1$

$= (3x^2 - 2x + 1)(x^2 + 1)$

remaining zeros are $\frac{1}{3}-\frac{\sqrt{2}}{3}i, i, -i$

2 **a** $f(-1) = 0$

$\therefore -1 + 13 + a = 0$

$\therefore a = -12$

$f(x) = x^3 - 13x - 12$

$= (x + 1)(x^2 - x - 12)$

$= (x + 1)(x - 4)(x + 3)$

remaining zeros are -3 and 4

b $f(3) = 0 \quad \therefore 27 - 63 + 3a - 15 = 0$

$\therefore a = 17$

$f(x) = x^3 - 7x^2 + 17x - 15$

$= (x - 3)(x^2 - 4x + 5)$

remaining zeros are $\frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

c $f(-1 - i) = 0, (-1 - i)^4 + 2(-1 - i)^3$
$- 2(-1 - i)^2 - 8(-1 - i) + a = 0$

$-4 + 2\,(2 - i) - 2(2i) - 8(-1 - i) + a = 0$

$-4 + 4 - 4i - 4i + 8 + 8i + a = 0$

$\therefore a = -8$

$f(x) = x^4 + 2x^3 - 2x^2 - 8x - 8$

$(x - (-1 - i))(x - (-1 + i)) = x^2 + 2x + 2$

$f(x) = (x^2 + 2x + 2)(x^2 - 4)$

$= (x - (-1 - i))(x - (-1 + i))(x - 2)(x + 2)$

remaining zeros are $-1 + i, 2, -2$

d $f(-2i) = 0 \quad (-2i)^4 - 4(-2i)^3 + 9(-2i)^2 +$
$a(-2i) + b = 0$

$16 - 32i - 36 - 2ai + b = 0$

$-32 - 2a = 0 \qquad -20 + b = 0$

$a = -16 \qquad b = 20$

$f(x) = x^4 - 4x^3 + 9x^2 - 16x + 20$

$(x - 2i)(x + 2i) = x^2 + 4$

$f(x) = (x^2 + 4)(x^2 - 4x + 5)$

remaining zeros are $2i, 2 + i, 2 - i$

Exercise 3N

1 $3x^3 - 2x^2 - 5x - 4 = 0$

a $x_1 + x_2 + x_3 = \frac{2}{3}$

b $x_1 \cdot x_2 \cdot x_3 = \frac{4}{3}$

c $x_1 x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 = -\frac{5}{3}$

d $\frac{6}{x_1} + \frac{6}{x_2} + \frac{6}{x_3} = \frac{6\left(x_2 x_3 + x_1 x_3 + x_1 x_2\right)}{x_1 x_2 x_3} = \frac{6\left(\frac{-5}{3}\right)}{\frac{4}{3}} = -\frac{15}{2}$

e $9x_1{}^2 + 9x_2{}^2 + 9x_3{}^2 = 9[(x_1 + x_2 + x_3)^2$
$-2(x_1 x_2 + x_1 x_3 + x_2 x_3)]$

$= 9\left[\left(\frac{2}{3}\right)^2 - 2\left(\frac{-5}{3}\right)\right] = 9\left(\frac{4}{9} + \frac{10}{3}\right)$

$= 34$

2 $x^4 - 3x^3 + 2x^2 - 4x - 6 = 0$

a $x_1 + x_2 + x_3 + x_4 = 3$

b $x_1 \cdot x_2 \cdot x_3 \cdot x_4 = -6$

c $x_1 \cdot x_2 + x_1 \cdot x_3 + x_1 \cdot x_4 + x_2 \cdot x_3 + x_2 \cdot x_4 + x_3 \cdot x_4 = 2$

d $x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_4 + x_1 \cdot x_3 \cdot x_4 + x_2 \cdot x_3 \cdot x_4 = 4$

e $\frac{3}{x_1} + \frac{3}{x_2} + \frac{3}{x_3} + \frac{3}{x_4} = \frac{3(4)}{-6} = -2$

f $(x_1 + x_2 + x_3 + x_4)^2 = x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 +$
$2(x_1 \cdot x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4)$

$\therefore \frac{x_1^2}{5} + \frac{x_2^2}{5} + \frac{x_3^2}{5} + \frac{x_4^2}{5} = \frac{1}{5}\left[3^2 - 2(2)\right] = 1$

3 **a** $f(x) = x^4 + 2x^3 - 3x^2 + 4x + 5$

sum $= -\frac{2}{1} = -2$ product $= (-1)^4 \frac{5}{1} = 5$

b $f(x) = 4x^6 + x^5 + 7x^4 - 3x^3 + 2x$

sum $= -\frac{1}{4}$ product $= 0$

c $f(x) = 11x^{10} - \frac{3}{7}x^7 + \sqrt{5}x^3 - \pi x + 22$

sum $= 0$ product $= (-1)^{10}\frac{22}{11} = 2$

d $f(x) = 5x^{7007} - 4x^{7006} + 2x^{231} + 10x + 8$

sum $= \frac{4}{5}$ product $= -\frac{8}{5}$

Exercise 3O

1 **a** $x^3 - 6x^2 + 11x - 6 = 0$

$(x - 1)(x^2 - 5x + 6) = 0$

$(x - 1)(x - 2)(x - 3) = 0$

$x = 1, 2$ or 3

b $x^3 + 2x^2 - 7x + 4 = 0$

$(x - 1)(x^2 + 3x - 4) = 0$

$(x - 1)(x - 1)(x + 4) = 0$

$x = 1, 1, -4$

c $x^3 + 3x^2 - 4x - 12 = 0$

$(x - 2)(x^2 + 5x + 6) = 0$

$(x - 2)(x + 2)(x + 3) = 0$

$x = 2, -2, -3$

d $2x^3 - 5x^2 - 18x + 45 = 0$

$(x - 3)(2x^2 + x - 15) = 0$

$(x - 3)(2x - 5)(x + 3) = 0$

$x = 3, \frac{5}{2}, -3$

Exercise 3P

1 **a** $12x^3 + 17x^2 + 2x - 3 = 0$

$(x + 1)(12x^2 + 5x - 3) = 0$

$(x - 1)(3x - 1)(4x + 3) = 0$

$x = -1, \frac{-3}{4}, \frac{1}{3}$

b $x^3 - 4x^2 - 5x + 14 = 0$

$(x + 2)(x^2 - 6x + 7) = 0$

$x = -2$ or $x = \frac{6 \pm \sqrt{36-28}}{2} = \frac{6 \pm \sqrt{8}}{2} = 3 \pm \sqrt{2}$

$x = -2, 3 \pm \sqrt{2}$

c $3x^3 - 13x^2 + 11x + 14 = 0$

$(3x + 2)(x^2 - 5x + 7) = 0$

$x = \frac{-2}{3}$

d $x^4 - x^3 - 11x^2 + 9x + 18 = 0$

$(x + 1)(x - 2)g(x) = 0$

$(x^2 - x - 2)(x^2 - 9) = 0$

$(x + 1)(x - 2)(x - 3)(x + 3) = 0$

$x = -1, 2, 3, -3$

2 $x^3 + ax^2 - x - 3 = 0$

a $-27 + 9a + 3 - 3 = 0 \quad \therefore 9a = 27$

$a = 3$

b $x^3 + 3x^2 - x - 3 = 0$

$(x + 3)(x^2 - 1) = 0$

$(x + 3)(x - 1)(x + 1) = 0$

other roots are 1, −1

3 $ax^3 - 7x^2 + bx + 4 = 0$

a $(x - 2)(x - 2)(ax + 1) = 0$

$(x^2 - 4x + 4)(ax + 1) = 0$

$ax^3 + x^2 - 4ax^2 - 4x + 4ax + 4 = 0$

$ax^3 + (1 - 4a)\,x^2 + (4a - 4)x + 4 = 0$

$1 - 4a = -7 \qquad \therefore a = 2$

$4a - 4 = b \qquad\quad b = 4$

b remaining root $= -\frac{1}{a} = -\frac{1}{2}$

4 Let $x = a$ be an integer zero

$\therefore a^3 + 5a + p = 0$

$p = -a\,(a^2 + 5)$

$\therefore$ a and $a^2 + 5$ are factors of p

$\therefore$ p is not prime

$\therefore$ If p is prime there are no integer zeros

5 $f(x) = x^3 + ax^2 + bx + c$

a Let the 2 zeros be p, $-p$

$$f(x) = (x - p)(x + p)\left(x - \frac{c}{p^2}\right)$$

$$= (x^2 - p^2)\left(x - \frac{c}{p^2}\right)$$

$$= x^3 - \frac{c}{p^2}x^2 - p^2x + c$$

$\therefore a = -\frac{c}{p^2} \qquad b = -p^2$

$\therefore \quad ab = \left(-\frac{c}{p^2}\right)(-p^2) = c \qquad\qquad \therefore ab = c$

b third zero $= \frac{c}{p^2} = \frac{-c}{b} = -a$

Exercise 3Q

1 **a** $x^3 - 6x^2 + 11x - 6 \ge 0$

$(x - 1)(x^2 - 5x + 6) \ge 0$

$(x - 1)(x - 2)(x - 3) \ge 0$

x	**]−∞, 1[**	**]1, 2[**	**]2, 3[**	**]3, ∞[**
x − 1	−	+	+	+
x − 2	−	−	+	+
x − 3	−	−	−	+
f(x)	−	+	−	+

$\therefore x \in [1, 2] \cup [3, \infty[$

b $x^3 + 2x^2 - 7x + 4 \le 0$

$(x - 1)(x^2 + 3x - 4) \le 0$

$(x - 1)(x - 1)(x + 4) \le 0$

x	**]−∞, −4[**	**]−4, 1[**	**]1, ∞[**
(x − 1)²	+	+	+
(x + 4)	−	+	+
f(x)	−	+	+

$\therefore x \in \,]-\infty, -4]$ or $x = 1$

c $x^3 + 3x^2 - 4x - 12 < 0$

$(x - 2)(x^2 + 5x + 6) < 0$

$(x - 2)(x + 2)(x + 3) < 0$

x	**]−∞, −3[**	**] −3, −2[**	**] −2, 2[**	**]3, ∞[**
x − 2	−	−	−	+
x + 2	−	−	+	+
x + 3	−	+	+	+
f(x)	−	+	−	+

$\therefore x \in \,]-\infty, -3\,[\cup]\,-2, 2[$

d $2x^3 - 5x^2 - 18x + 45 > 0$

$(x - 3)(2x^2 + x - 15) > 0$

$(x - 3)(2x - 5)(x + 3) > 0$

x	**]−∞, −3[**	**]−3, 2.5[**	**]2.5, 3[**	**]3, ∞[**
x − 3	−	−	−	+
2x − 5	−	−	+	+
x + 3	−	+	+	+
f(x)	−	+	−	+

$\therefore x \in \,]-3, 2.5\,[\cup]\,3, \infty[$

e $12x^3 + 17x^2 + 2x - 3 \le 0$

$(x + 1)(12x^2 + 5x - 3) \le 0$

$(x + 1)(4x + 3)(3x - 1) \le 0$

x	**]−∞, −1[**	**]−1, $\frac{-3}{4}$[**	**]$\frac{-3}{4}$, $\frac{1}{3}$[**	**]$\frac{1}{3}$, ∞[**
x + 1	−	+	+	+
4x + 3	−	−	+	+
3x − 1	−	−	−	+
f(x)	−	+	−	+

$\therefore x \in \,]-\infty, -1] \cup [\frac{-3}{4}, \frac{1}{3}]$

f $x^3 - 4x^2 - 5x + 14 > 0$

$(x + 2)(x^2 - 6x + 7) > 0 \quad \frac{6 \pm \sqrt{36-28}}{2} = 3 \pm \sqrt{2}$

x	$]-\infty, -2[$	$]-2, 3-\sqrt{2}[$	$]3-\sqrt{2}, 3+\sqrt{2}[$	$]3+\sqrt{2}, \infty[$
$x + 2$	–	+	+	+
$x - (3+\sqrt{2})$	–	–	–	+
$x - (3-\sqrt{2})$	–	–	+	+
$f(x)$	–	+	–	+

$x \in]-2, 3-\sqrt{2}[\; \cup \;]3+\sqrt{2}, \infty[$

g $3x^3 - 13x^2 + 11x + 14 < 0$

$(3x + 2)(x^2 - 5x + 7) < 0$

$x = \frac{-2}{3}$

x	$]-\infty, \frac{-2}{3}[$	$]\frac{-2}{3}, \infty[$
$3x + 2$	–	+
$x^2 - 5x + 7$	+	+
$f(x)$	–	+

$x \in]-\infty, \frac{-2}{3}[$

h $x^4 - x^3 - 11x^2 + 9x + 18 \geq 0$

$(x + 1)(x - 2)\, g(x) \geq 0$

$(x^2 - x - 2)(x^2 - 9) \geq 0$

$(x + 1)(x - 2)(x - 3)\,(x + 3) \geq 0$

x	$]-\infty, -3[$	$]-3, -1[$	$]-1, 2[$	$]2, 3[$	$]3, \infty[$
$x + 1$	–	–	+	+	+
$x - 2$	–	–	–	+	+
$x - 3$	–	–	–	–	+
$x + 3$	–	+	+	+	+
$f(x)$	+	–	+	–	+

$\therefore x \in]-\infty, -3] \cup [-1, 2] \cup [3, \infty[$

2 $f(x) > g(x)$

$4x^3 - 17x^2 + 30x + 5 > -2x^3$

$+ 8x^2 + 9x - 5$

$6x^3 - 25x^2 + 21x + 10 > 0$

$(x - 2)(6x^2 - 13x - 5) > 0$

$(x - 2)(3x + 1)(2x - 5) > 0$

x	$]-\infty, -\frac{1}{3}[$	$]-\frac{1}{3}, 2[$	$]2, \frac{5}{2}[$	$]\frac{5}{2}, \infty[$
$x - 1$	–	–	+	+
$3x + 1$	–	+	+	+
$2x - 5$	–	–	–	+
$f(x) - g(x)$	–	+	–	+

$x \in]-\frac{1}{3}, 2[\; \cup \;]\frac{5}{2}, \infty[$

3 **a** $x^7 - 2x^2 - 1 \geq 0$

$x \in [-1, -0.921] \cup [1.26, \infty[$

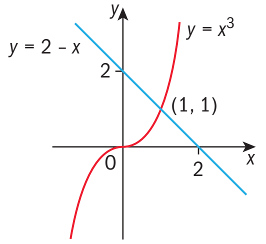
b $x^9 - 2x^8 + 2x^5 + x \leq 0$

$x \in]-\infty, 0]$

4 **a** $x^3 + x - 2 > 0$

$x^3 > 2 - x$

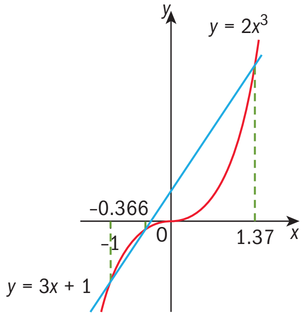
$x \in]1, \infty[$



b $-2x^3 + 3x + 1 \geq 0$

$3x + 1 \geq 2x^3$

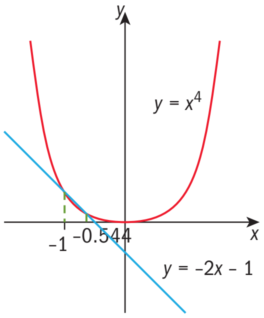
$x \in]-\infty, -1] \cup [-0.366, 1.37]$



c $x^4 + 2x + 1 \leq 0$

$x^4 \leq -2x - 1$

$x \in [-1, -0.544]$



Exercise 3R

1 **a** $2ix + (2 + 3i)y = i \qquad a = 2i \quad b = 2+ 3i \quad e = i$

$(1 + i)x + 2y = 3 \qquad c = 1 + i \quad d = 2 \quad f = 3$

$$D = \begin{vmatrix} 2i & 2+3i \\ 1+i & 2 \end{vmatrix} = 4i - (1 + i)(2 + 3i)$$

$$= 4i - 2 - 5i + 3 = 1 - i$$

$$D_x = \begin{vmatrix} i & 2+3i \\ 3 & 2 \end{vmatrix} = 2i - 3\,(2 + 3i) = -6 - 7i$$

$$D_y = \begin{vmatrix} 2i & i \\ 1+i & 3 \end{vmatrix} = 6i - i\,(1 + i) = 5i + 1$$

$$x = \frac{-6-7i}{1-i} \times \frac{1+i}{1+i} = \frac{-6-6i-7i+7}{2} = \frac{1}{2} - \frac{13}{2}i$$

$$y = \frac{1+5i}{1-i} \times \frac{1+i}{1+i} = \frac{1+i+5i-5}{2} = -2 + 3i$$

b $(1 + i)x + 3iy = 2 + 6i \quad a = 1 + i \quad b = 3i$
$e = 2 + 6i$
$(2 - i)\,x - (4 + 3i)\,y = 4i - 3 \quad c = 2 - i$
$d = -4 -3i \quad f = -3 + 4i$

$$D = \begin{vmatrix} 1+i & 3i \\ 2-i & -4-3i \end{vmatrix} = (1 + i)(-4 -3i) - 3i\,(2 - i)$$

$$= 4 - 3i - 4i + 3 - 6i - 3 = -4 -13i$$

$$D_x = \begin{vmatrix} 2+6i & 3i \\ -3+4i & -4-3i \end{vmatrix} = (2 + 6i)(-4 -3i)$$

$$-3i(-3 + 4i)$$

$$= -8 -6i -24i + 18 + 9i + 12 = 22 - 21i$$

$$D_y = \begin{vmatrix} 1+i & 2+6i \\ 2-i & -3+4i \end{vmatrix} = (1 + i)(-3 + 4i) - (2 + 6i)$$

$$(2 - i)$$

$$= -3 + 4i - 3i - 4 - (4 - 2i + 12i + 6)$$
$$= -17 -9i$$

$$x = \frac{22 - 21i}{-4 - 13i} \times \frac{-4 + 13i}{-4 + 13i} = \frac{-88 + 286i + 84i + 273}{185} = 1 + 2i$$

$$y = \frac{-17 - 9i}{-4 - 13i} \times \frac{-4 + 13i}{-4 + 13i} = \frac{68 - 221i + 36i + 117}{185} = 1 - i$$

2 **a** $x + y = -1 \quad (1) \qquad x + y = -1 \quad (1)$
$x + z = 4 \quad (2) \quad (2) - (3) \quad x - y = 3 \quad (4)$
$y + z = 1 \quad (3)$
$(1) + (4) \quad 2x = 2 \quad \therefore x = 1$
sub in (1) $\quad 1 + y = -1 \quad y = -2$
sub in (3) $\quad -2 + z = 1 \quad z = 3 \quad (1, -2, 3)$

b $x - 5y + 3z = -1$ (1) (1) + 3(3) $7x - 2y = 5$ (4)
$3x - y + 2z = 4$ (2) (2) + 2(3) $7x + y = 8$ (5)
$2x + y - z = 2$ (3)
(5) − (4) $\quad 3y = 3, y = 1$
sub in (5) $\quad 7x + 1 = 8 \quad x = 1$
sub in (3) $\quad 2 + 1 - z = 2 \quad z = 1 \quad (1, 1, 1)$

c $2x + y + 2z = 0$ (1) 4(1) + (2): $14x + 3z = -2$ (4)
$6x - 4y - 5z = -2$ (2) (3) − (1): $2x - 5z = 2$ (5)
$4x + y - 3z = 2$ (3)
(4) − 7(5) $\quad 38z = -16 \quad z = \frac{-8}{19}$
sub in (5) $\quad 2x + \frac{40}{19} = 2, \quad x = \frac{-1}{19}$
sub in (1) $\quad \frac{-2}{19} + y - \frac{16}{19} = 0 \quad y = \frac{18}{19} \quad \left(\frac{-1}{19}, \frac{18}{19}, \frac{-8}{19}\right)$

d $3x - 4y + 3z = -2$ (1) (1) + (3) $5x - 10\,y = -10$
$x + 2y + 6z = 6$ (2) $\quad x - 2y = -2 \quad (4)$
$2x - 6y - 3z = -8$ (3) (2) + 2(3) $5x - 10y = -10$
$x - 2y = -2$ (5)
(4) + (5) are the same equation ∴ infinite number of solutious $\quad x = 2y - 2$
sub in (1) $6y - 6 - 4y + 3z = -2, 3x = 4 - 2y,$
$z = \frac{4 - 2y}{3}$
$\left(2y - 2,\ y,\ \frac{4 - 2y}{3}\right)$

e $x + 2y + z = 4$ (1) $\quad$ (2) − 2(1) $-3y = -3, y = 1$
$2x + y + 2z = 5$ (2)
$3x + 2y + 3z = 12$ (3)
sub in (2) $\quad 2x + 2z = 4$ } inconsistent
sub in (3) $\quad 3x + 3z = 10$ } ∴ no solution

f $2x - 3y + 5z = -1$ (1) (1) − 2(3) $y - z = -19$ (4)
$9x - 7y + 16z = 0$ (2) (2) − 9(3) $11y - 11z = -81$ (5)
$x - 2y + 3z = 9$ (3)
(4) + (5) are inconsistent ∴ no solution

3 **a** $x + 2y + z = 0$ (1) (1) − (3) $z - kz = -2$
$2x + y + 2z = 1$ (2) $\quad (1 - k)\,z = -2$
$x + 2y + kz = 2$ (3) no solution if $k = 1$
If $k \neq 1,\ z = \frac{-2}{1 - k}$
sub in (1) $x + 2y = \frac{2}{1 - k}$ (4)
sub in (2) $2x + y = 1 + \frac{4}{1 - k} = \frac{5 - k}{1 - k}$ (5)
(5) − 2(4) $-3y = \frac{5 - k}{1 - k} - \frac{4}{1 - k}$
$-3y = \frac{1 - k}{1 - k} \quad \therefore \quad y = -\frac{1}{3}$
sub in (4) $x = \frac{2}{1 - k} + \frac{2}{3} = \frac{6 + 2(1 - k)}{3(1 - k)} = \frac{8 - 2k}{3(1 - k)}$
∴ no unique solution if $k = 1$

b $x + y + z = 1$ (1) (2) − 2(1) $(k - 2)\,y + z = -4$ (4)
$2x + ky + 3z = -2$ (2) (3) − 3(1) $2y + (k - 3)\,z = -4$ (5)
$3x + 5y + kz = -1$ (3)
For no unique solution
$\frac{k - 2}{2} = \frac{1}{k - 3}$
$(k - 2)(k - 3) = 2$
$k^2 - 5k + 4 = 0$
$(k - 4)(k - 1) = 0$
$k = 1$ or 4

4 **a** $x + 2y + 3z = 1$ (1) (1) − (2) $(1 - k)\,x - 2y$
$= -1$ (4)

$kx + 4y + 3z = 2$ (2) 2(1) + 3(3) $11x + 22y$
$= 11$

$3x + 6y - 2z = 3$ (3) $x + 2y = 1$ (5)

For an infinite number of solutious

$1 - k = -1,\ k = 2$

$x = 1 - 2y$

sub in (3) $2z = 3x + 6y - 3$

$= 3(1 - 2y) + 6y - 3$

$\therefore\ z = 0$

$(1 - 2y, y, 0)$

b $x + y + z = 1$ (1)

$2x + ky + 3z = -2$ (2)

$3x + 5y + kz = -1$ (3)

From 3**b**, $(k - 2)\,y + z = -4$ (4)

$2y + (k - 3)\,z = -4$ (5)

For no unique solution $k = 1$ or 4

If $k = 1$: $\left.\begin{matrix} -y+z=-4 \\ 2y-2z=-4 \end{matrix}\right\}$ no solution

If $k = 4$: $\left.\begin{matrix} 2y+z=-4 \\ 2y+z=-4 \end{matrix}\right\}$ infinitely many solutious

$z = -4 - 2y$

sub in (1) $x = 1 - y - (-4 - 2y) = 5 + y$

$(y + 5, y, -4 - 2y)$

5 $x + y + z = m$ (1) (1) − (3) $z - mz = m + 1$

$x + my + z = 2m$ (2) $z(1 - m) = m + 1$

$x + y + mz = -1$ (3) unique solution if $m \neq 1$

$$z = \frac{1+m}{1-m}$$

sub in (1) $x + y + \dfrac{1+m}{1-m} = m$

$$x + y = m - \frac{1+m}{1-m} = \frac{m - m^2 - 1 - m}{1-m}$$

$$x + y = \frac{-m^2 - 1}{1-m} \quad (4)$$

sub in (2) $x + my = 2m - \dfrac{1+m}{1-m} = \dfrac{2m - 2m^2 - 1 - m}{1-m}$

$$x + my = \frac{m - 2m^2 - 1}{1-m} \quad (5)$$

(4) − (5): $y(1-m) = \dfrac{-m^2 - 1 - m + 2m^2 + 1}{1-m}$

$$\frac{m^2 - m}{1-m} = \frac{m(m-1)}{1-m} = -m$$

$$\therefore \quad y = \frac{-m}{1-m}$$

sub in (4) $x = \dfrac{-m^2 - 1}{1-m} + \dfrac{m}{1-m} = \dfrac{m - m^2 - 1}{1-m}$

$$\left(\frac{m - m^2 - 1}{1-m}, \frac{-m}{1-m}, \frac{1+m}{1-m}\right)$$

Review exercise

1 $f(x) = x^4 - 3x^3 + ax^2 - 4x + 7$

$f(-2) = 7 \quad \therefore\ 16 + 24 + 4a + 8 + 7 = 0$

$4a = -55$

$$a = \frac{-55}{4}$$

2 $3x - 2y = i - 2$

$4y - (1 - i)\,x = 3 + 3i \Rightarrow -(1 - i)\,x + 4y = 3 + 3i$

$a = 3 \quad b = -2 \quad c = -1 + i \quad d = 4 \quad e = -2 + i$

$f = 3 + 3i$

$$D = \begin{vmatrix} 3 & -2 \\ -1+i & 4 \end{vmatrix} = 12 + 2\,(-1 + i) = 10 + 2i$$

$$D_x = \begin{vmatrix} -2+i & -2 \\ 3+3i & 4 \end{vmatrix} = 4(-2 + i) + 2(3 + 3i) = -2 + 10i$$

$$D_y = \begin{vmatrix} 3 & -2+i \\ -1+i & 3+3\text{i} \end{vmatrix} = 3(3 + 3i) - (-1 + i)(-2 + i)$$

$= 9 + 9i - (2 - 3i - 1) = 8 + 12i$

$$x = \frac{-2 + 10i}{10 + 2i} \times \frac{10 - 2i}{10 - 2i} = \frac{-20 + 4i + 100i + 20}{104} = i$$

$$y = \frac{8 + 12i}{10 + 2i} \times \frac{10 - 2i}{10 - 2i} = \frac{80 - 16x + 120i + 24}{104} = 1 + i$$

$x = i,\ y = 1 + i$

3 $f(x) = m - 2 + (2m + 1)\,x + mx^2 \quad m < 0$

$b^2 - 4ac < 0$

$(2m + 1)^2 - 4\,(m - 2)\,m < 0$

$4m^2 + 4m + 1 - 4m^2 + 8m < 0$

$12m + 1 < 0$

$$m < \frac{-1}{12}$$

4 $z^4 - 2z^3 + 14z^2 - 18z + 45 = 0$

$(z - (1 - 2i))(z - (1 + 2i)) = z^2 - 2z + 5$

$(z^2 - 2z + 5)(z^2 + 9) = 0$

remaining roots are $1 + 2i, -3i, 3i$

5 $mx + 2y = 1$

$4x + (m + 2)\,y = 4$ no unique solution

$\dfrac{m}{4} = \dfrac{2}{m+2} \qquad m^2 + 2m = 8$

$m^2 + 2m - 8 = 0$

$(m + 4)(m - 2) = 0$

$m = -4$ or 2

6 $x^2 + ax + a + 1 = 0$

$\alpha + \beta = -a$

$\alpha\beta = a + 1$

$(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$(\alpha + \beta)^3 = (\alpha^3 + \beta^3) + 3\alpha\beta(\alpha + \beta)$

$-a^3 = 9 - 3a(a + 1)$

$a^3 - 3a^2 - 3a + 9 = 0$

$(a - 3)(a^2 - 3) = 0$

$a = 3, \pm\sqrt{3}$

7 P_n: $z^{2n} = \dfrac{i^n}{2^n}$, $\quad n \in \mathbb{Z}^+$

Let $n = 1$, $z^2 = \dfrac{(1+i)^2}{2^2} = \dfrac{1 + 2i - 1}{4} = \dfrac{i}{2}$

$\therefore$ P_1 is true

Assume P_k: $z^{2k} = \dfrac{i^k}{2^k}$

Prove P_{k+1}: $z^{2(k+1)} = z^2 z^{2\text{k}} = \dfrac{i}{2}\left(\dfrac{i^k}{2^k}\right) = \dfrac{i^{k+1}}{2^{k+1}}$

$\therefore \text{P}_k \Rightarrow \text{P}_{k+1}$

$\therefore$ by mathematical induction, $z^{2n} = \dfrac{i^n}{2^n}$

8 $\dfrac{1+i}{1-i} = \dfrac{1+i}{1-i} \times \dfrac{1+i}{1+i} = \dfrac{1 + 2i - 1}{2} = i$

$\left(\dfrac{1+i}{1-i}\right)^{2011} = i^{2011} = i^{2008}\, i^3 = -i$

$\therefore$ imaginary part $= -1$

9 $x^3 - 5x^2 + 6x - 3 = 0$

$\alpha + \beta + \gamma = 5$

$\alpha\beta + \alpha\gamma + \beta\gamma = 6 \quad \alpha\beta\gamma = 3$

$\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2} = \dfrac{\alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2}{\alpha^2\beta^2\gamma^2}$

$(\alpha\beta + \beta\gamma + \alpha\gamma)^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2$

$\qquad + 2(\alpha\beta^2\gamma + \alpha^2\beta\gamma + \alpha\beta\gamma^2)$

$= \alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha\beta\gamma(\beta + \alpha + \gamma)$

$\therefore \alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2 = 6^2 + 2(3)(5) = 66$

$\therefore \dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2} = \dfrac{6}{3^2} = \dfrac{6}{9} = \dfrac{2}{3}$

10 **a** Let $a = \sqrt[3]{7 - \sqrt{50}}$, $b = \sqrt[3]{7 + \sqrt{50}}$, $x = a + b$

$x^3 + 3x - 14 = (a + b)^3 + 3(a + b) - 14$

$= a^3 + 3a^2b + 3ab^2 + b^3 + 3(a + b) - 14$

$= a^3 + b^3 + 3ab(a + b) + 3(a + b) - 14$

$a^3 + b^3 + 3(a + b)(ab + 1) - 14$

$ab = \sqrt[3]{\left(7 - \sqrt{50}\right)\left(7 + \sqrt{50}\right)} = \sqrt[3]{49 - 50} = \sqrt[3]{-1} = -1$

$\therefore x^3 + 3x - 14 = a^3 + b^3 + 3(a + b)(-1 + 1) - 14$

$= a^3 + b^3 - 14 = 0$

$= 7 - \sqrt{50} + 7 + \sqrt{50} - 14 = 0$

$\therefore \sqrt[3]{7 - \sqrt{50}} + \sqrt[3]{7 + \sqrt{50}}$ satisfies the equation $x^3 + 3x - 14$

b $f(x) = z^3 + 3z - 14$

$= (z - 2)(z^2 + 2z + 7)$

$z = 2$ or $z = \dfrac{-2 \pm \sqrt{4 - 28}}{2} = -1 \pm i\sqrt{6}$

$z = 2, -1 \pm i\sqrt{6}$

c since 2 is the only real root,

$\sqrt[3]{7 - \sqrt{50}} + \sqrt[3]{7 + \sqrt{50}} = 2$

Review exercise

1 $x \in [1.67, \infty[$

2 $(mx)^2 + 3x + 1 - m = 0$

$b^2 - 4ac < 0$

$9 - 4m^2(1 - m) < 0$

$9 - 4m^2 + 4m^3 < 0$

$m \in\ ]-\infty, -1.05[$

3 $2x + 14y + 9z = -7 \qquad 2x + 14y + 9z = -7$

$4x - 3z = 4 + 7y \qquad 4x - 7y - 3z = 4$

$10x - 28y = 5 + 6z \qquad 10x - 28y - 6z = 5$

$x = \dfrac{1}{2}, y = \dfrac{2}{7}, z = \dfrac{-4}{3}$

4 $3x^3 + 2x = 5x^2 + 4$

$3x^3 - 5x^2 + 2x - 4 = 0$

$\alpha + \beta + \gamma = \dfrac{5}{3}$, $\alpha\beta + \beta\gamma + \alpha\gamma = \dfrac{2}{3}$, $\alpha\beta\gamma = \dfrac{4}{3}$

$(\alpha + \beta + \gamma)^3 = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\alpha\gamma)$

$= \alpha^3 + \beta^3 + \gamma^3 + 6\alpha\beta\gamma + 3(\alpha\beta^2 + \alpha\gamma^2 + \alpha^2\beta + \alpha^2\gamma + \beta\gamma^2 + \beta^2\gamma)$

$\alpha^3 + \beta^3 + \gamma^3 = \dfrac{125}{27} - 6\left(\dfrac{4}{3}\right) - 3(\alpha\beta^2 + \alpha\gamma^2 + \alpha^2\beta + \alpha^2\gamma + \beta\gamma^2 + \beta^2\gamma)$

$(\alpha + \beta + \gamma)(\gamma\beta + \beta\gamma + \alpha\gamma) = 3\alpha\beta\gamma + \alpha\beta^2 + \alpha\gamma^2 + \alpha^2\beta + \alpha^2\gamma + \beta\gamma^2 + \beta^2\gamma$

$\therefore \alpha\beta^2 + \alpha\gamma^2 + \alpha^2\beta + \alpha^2\gamma + \beta\gamma^2 + \beta^2\gamma$

$= \dfrac{5}{3}\left(\dfrac{2}{3}\right) - 3\left(\dfrac{4}{3}\right) = \dfrac{-26}{9}$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = \dfrac{125}{27} - 6\left(\dfrac{4}{3}\right) - 3\left(\dfrac{-26}{9}\right)$

$= \dfrac{143}{27}$

5 $f(x) = x^7 + 35x^6 - 97x^5 + 33x^2 + 4$

smallest zero $= 0.833$

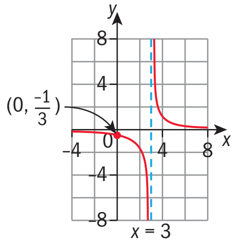
4

Modeling the real world

Answers

Skills check

1



2 $\sum_{n=0}^{\infty} 5\left(\frac{1}{2}\right)^n = \frac{5}{1-\frac{1}{2}} = 10$

Exercise 4A

1 $\lim_{x\to -1} \frac{x^2-1}{x+1} = -2$

2 $\lim_{x\to 1} \frac{x^3-1}{x-1} = 3$

3 $\lim_{x\to 2^+} f(x) = \frac{1}{3} \qquad \lim_{x\to 2^-} f(x) = 5$

$\therefore \lim_{x\to 2} \begin{cases} 3x-1 & x<2 \\ \frac{1}{x^2-1} & x\geq 2 \end{cases}$ does not exist

4 $\lim_{x\to 0^-} \frac{|x|}{x} = -1 \quad \lim_{x\to 0^+} \frac{|x|}{x} = 1 \quad \therefore \lim_{x\to 0} \frac{|x|}{x}$ does not exist

5 $\lim_{x\to 6} (x-6)^{\frac{2}{3}} = 0$

6 $\lim_{x\to 3^-} [x] = 2 \qquad \lim_{x\to 3^+} [x] = 3$

$\therefore \lim_{x\to 3} [x]$ does not exist.

Exercise 4B

1 $\lim_{x\to 1^-} f(x) = 0 \qquad \lim_{x\to 1^+} f(x) = 2$

$\therefore \lim_{x\to 1} f(x)$ does not exist

$\therefore f$ is not continuous at $x = 1$

2 $\lim_{x\to -2^-} f(x) = 1 \qquad \lim_{x\to -2^+} f(x) = 1 \qquad \therefore \lim_{x\to -2} f(x) = 1$

Also, $f(-2) = 1 \qquad \therefore f$ is continuous at $x = -2$

3 $\lim_{x\to 1^-} f(x) = -1 \qquad \lim_{x\to 1^+} f(x) = 1$

$\therefore \lim_{x\to 1} f(x)$ does not exist

$\therefore f$ is not continuous at $x = 1$

4 $\lim_{x\to 3^-} f(x) = 8 \quad \therefore f(3) = 8 \quad \therefore 6 = 8 \quad \therefore k = \frac{4}{3}$

5 $\lim_{x\to 3^-} f(x) = 4 \quad \therefore f(3) = 4 \quad \therefore 9a - a = 4$

$8a = 4$

$a = \frac{1}{2}$

6 **a** discontinuous at $x = \pm 1$

b discontinuous at $x = \pm 2$

c continuous

d discontinuous at $x = -4$ and $x = 1$

e discontinuous at $x = 1$

f continuous

Exercise 4C

1 **a** $\lim_{x\to 4} \left(\frac{x+3}{x-3}\right) = 7$

b $\lim_{x\to -2} \left(\frac{x^2+x-2}{x+2}\right) = \lim_{x\to -2} \left(\frac{(x+2)(x-1)}{(x+2)}\right)$

$= \lim_{x\to -2} (x-1) = -3$

c $\lim_{x\to -2} \left(\frac{x^6-64}{x^3-8}\right) = \lim_{x\to -2} \left(\frac{(x^3+8)(x^3-8)}{x^3-8}\right)$

$= \lim_{x\to -2} (x^3+8) = 0$

d $\lim_{x\to 0} \left(\frac{x^2-1}{x^2-x}\right) = \lim_{x\to 0} \left(\frac{(x-1)(x+1)}{x(x-1)}\right)$

$= \lim_{x\to 0} \left(\frac{x+1}{x}\right) = \lim_{x\to 0} \left(1+\frac{1}{x}\right),$

which does not exist

e $\lim_{x\to 1} \frac{x^2-1}{x^2-x} = \lim_{x\to 1} \left(1+\frac{1}{x}\right) = 2$

f $\lim_{x\to 1} \frac{1}{1+\frac{1}{1-x}} = \lim_{x\to 1} \frac{1-x}{2-x} = 0$

g $\lim_{x\to 0} \frac{(2+3x)^2 - 4(1+x)^2}{6x}$

$= \lim_{x\to 0} \frac{4+12x+9x^2-4-8x-4x^2}{6x}$

$= \lim_{x\to 0} \frac{5x^2+4x}{6x} = \lim_{x\to 0} \frac{5x+4}{6} = \frac{2}{3}$

h $\lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} \frac{(x-a)(x+a)}{x-a}$

$= \lim_{x\to a} (x+a) = 2a$

2 **a** $\lim\limits_{x\to\infty} \frac{2x}{x+2} = 2$

b $\lim\limits_{x\to\infty} \frac{3x^2}{x^2-1} = 3$

c $\lim\limits_{x\to\infty} \frac{2x^2+x-1}{3x^2+5x-1} = \frac{2}{3}$

d $\lim\limits_{x\to\infty} \frac{5x^2}{4x^3+2} = 0$

e $\lim\limits_{x\to\infty} \frac{x-1}{x^2-3x+5} = 0$

f $\lim\limits_{x\to\infty} \frac{\sqrt{4x+3}+2\sqrt{1+x}}{\sqrt{x}} = 4$

3 **a** $y = 3$ **b** $y = \frac{1}{2}$ **c** $y = 0$

d $y = -1$ **e** no horizontal asymptote

Exercise 4D

1 **a** converges **b** converges **c** converges

d diverges **e** converges

2 **a** converges, $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2^n} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

b $\sum\limits_{n=1}^{\infty}\left(\frac{\pi}{3.14}\right)^n$ diverges since $\frac{\pi}{3.14} > 1$

c converges, $\sum\limits_{n=1}^{\infty} 5\left(\frac{1}{3}\right)^n = \frac{\frac{5}{3}}{1-\frac{1}{3}} = \frac{5}{2}$

d converges, $\sum\limits_{n=1}^{\infty} \frac{3}{10^n} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{1}{3}$

e converges, $\sum\limits_{n=1}^{\infty} \frac{2^n-3^n}{7^n} = \sum\limits_{n=1}^{\infty}\left(\frac{2}{7}\right)^n - \sum\limits_{n=1}^{\infty}\left(\frac{3}{7}\right)^n$

$= \frac{\frac{2}{7}}{\frac{5}{7}} - \frac{\frac{3}{7}}{\frac{4}{7}} = \frac{2}{5} - \frac{3}{4} = \frac{-7}{20}$

f converges, $\sum\limits_{n=1}^{\infty} 4(-0.6)^{n-1} = \frac{4}{1+0.6} = 2.5$

3 $u_1 = 35$ $r = 2^x$

a $-1 < 2^x < 1$ 2^x must be positive

$\therefore\ 0 < 2^x < 1$ $\therefore\ x < 0$

b $\frac{35}{1-2^x} = 40$ $\therefore\ 1-2^x = \frac{7}{8}$ $\therefore\ 2^x = \frac{1}{8}$ $\therefore\ x = -3$

4 $-1 < \frac{3x}{x+1} < 1$ $\therefore\ -0.25 < x < 0.5$

Exercise 4E

1 **a** $y = 2x^2 - 1$ $(x = 1)$

$\frac{\Delta y}{\Delta x} = \frac{[2(1+h)^2-1]-[2(1)^2-1]}{(1+h)-1} = \frac{(1+4h+2h^2)-1}{h}$

$= \frac{4h+2h^2}{h} = 4+2h$

gradient $= \lim\limits_{h\to 0}(4+2h) = 4$

b $y = \frac{2}{x}$ $(x = -2)$

$\frac{\Delta y}{\Delta x} = \frac{\frac{2}{-2+h}-\frac{2}{-2}}{(-2+h)-(-2)}\ \frac{\frac{2}{-2+h}+1}{h} = \frac{2-2+h}{h(-2+h)} = \frac{1}{-2+h}$

gradient $= \lim\limits_{h\to 0}\left(\frac{1}{-2+h}\right) = -\frac{1}{2}$

c $y = x^3$ $(x = 1)$

$\frac{\Delta y}{\Delta x} = \frac{(1+h)^3-1^3}{(1+h)-1} = \frac{1+3h+3h^2+h^3-1}{h} = 3+3h+h^2$

gradient $= \lim\limits_{h\to 0}(3+3h+h^2) = 3$

$\therefore$ gradient $= 3$

d $y = -x^2$ $(x = 1)$

$f'(x) = \lim\limits_{h\to 0}\frac{-(x+h)^2-(-x^2)}{h}$

$= \lim\limits_{h\to 0}\frac{(-x^2-2xh-h^2+x^2)}{h}$

$= \lim\limits_{h\to 0}(-2x-h)$

$= -2x$

$f'(1) = -2(1) = -2$

e $y = \frac{x}{x+1}$

$f'(x) = \lim\limits_{h\to 0}\frac{\frac{x+h}{x+h+1}-\frac{x}{x+1}}{h}$

$= \lim\limits_{h\to 0}\frac{(x+h)(x+1)-x(x+h+1)}{(x+h+1)(x+1)h}$

$= \lim\limits_{h\to 0}\frac{h}{(x+h+1)(x+1)h} = \lim\limits_{h\to 0}\frac{1}{(x+h+1)(x+1)}$

$= \frac{1}{(x+1)^2}$

$f'(0) = 1$

f $y = \frac{1}{x^2}$ $(x = 2)$

$f'(x) = \lim\limits_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \lim\limits_{h\to 0}\frac{x^2-(x+h)^2}{hx^2(x+h)^2}$

$= \lim\limits_{h\to 0}\frac{x^2-x^2-2xh-h^2}{hx^2(x+h)^2} = \lim\limits_{h\to 0}\frac{-2x-h}{x^2(x+h)s'}$

$= \frac{-2x}{x^4} = \frac{-2}{x^3}$

$f'(2) = \frac{-2}{2^3} = -\frac{1}{4}$

2 $f'(x) = -\frac{2}{x^3}$

$f'(x) = 2 \Rightarrow \frac{-2}{x^3} = 2$

$\Rightarrow x^3 = -1 \Rightarrow x = -1$ $\therefore$ point is $(-1, 1)$

3 $f'(x) = 4x - \frac{1}{x^2}$

$f'(x) = 3 \Rightarrow 4x - \frac{1}{x^2} = 3$

$\Rightarrow 4x^3 - 1 = 3x^2 \Rightarrow 4x^3 - 3x^2 - 1 = 0$

$\Rightarrow (x-1)(4x^2+x+1) = 0$

$\Rightarrow x = 1$ $\therefore$ point is $(1, 3)$

Exercise 4F

1 **a** $y = x^2 + 2x + 1 \quad f'(x) = 2x + 2 \quad f'(0) = 2$

b $y = x^3 - 1 \quad f'(x) = 3x^2 \quad f'(1) = 3$

c $y = \frac{2}{x} \quad f'(x) = \frac{-2}{x^2} \quad \therefore f'(3) = \frac{-2}{9}$

d $y = \sqrt{x-1} \quad f'(x) = \frac{1}{2\sqrt{x-1}} \quad \therefore f'(2) = \frac{1}{2}$

e $y = \sqrt{x+3}, \quad f'(x) = \frac{1}{2\sqrt{x+3}}, \quad f'(1) = \frac{1}{4}$

f $y = \frac{1}{\sqrt{x}}, \quad f'(x) = -\frac{1}{2\sqrt{x}^3}, \quad f'(4) = -\frac{1}{16}$

2 **a** $$\text{Average velocity} = \frac{x(a+h)-x(a)}{h}$$
$$= \frac{12-5(a+h)^2-12+5a^2}{h}$$
$$= -10a - 5h$$

b velocity $= \lim\limits_{h\to0}$ (average velocity) $= -10a$

Exercise 4G

1 $y = 9 - x^2 \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -2x$

a When $x = -1$, gradient $= 2$

b When $x = -1$, gradient $= 8$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$

$\therefore$ tangent is $y - 8 = 2(x + 1)$ i.e. $y = 2x + 10$

c Normal is $y - 8 = -\frac{1}{2}(x + 1)$ i.e. $y = -\frac{1}{2}x + \frac{15}{2}$

2 $y = \frac{1}{x-1} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{(x-1)^2} \quad \therefore \frac{-1}{(x-1)^2} = -1$

$\therefore (x - 1)^2 = 1 \quad x - 1 = \pm 1$

$x = 0$ or $2 \quad (0, -1), (2, 1)$

At $(0, -1) \quad y + 1 = -1(x - 0) \quad y = -x - 1$

At $(2, 1) \quad y - 1 = -1(x - 2) \quad y = -x + 3$

3 **a** $y = 4 - 3x - 3x^2 \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -3 - 6x$

$-3 - 6x = 0 \quad \therefore x = x - \frac{1}{2} \quad \left(\frac{-1}{2}, \frac{19}{4}\right)$

b $y = x^3 + 1 \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$

$3x^2 = 0 \quad \therefore x = 0 \quad (0, 1)$

c $y = \frac{1}{x} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{x^2} \quad \frac{-1}{x^2} \neq 0 \quad \therefore$ no points

d $y = x^2 - 3x \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 3$

$2x - 3 = 0 \quad \therefore x = \frac{3}{2} \quad \left(\frac{3}{2}, \frac{-9}{4}\right)$

e $y = \sqrt{x} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} \quad \frac{1}{2\sqrt{x}} \neq 0 \quad \therefore$ no points

4 $y = x + \frac{1}{x} \quad \frac{dy}{dx} = 1 - \frac{1}{x^2}$

At $(1, 2) \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 - 1 = 0$

Equation of tangent is $y = 2$
$\therefore$ normal is $x = 1$

Exercise 4H

1 **a** $y = 4 - x - 3x^2 \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -1 - 6x$

b $y = 2x^4 - 3x + 1 \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 8x^2 - 3$

c $y = 4x^3 - \frac{1}{x^3} + 2x^2 + \frac{2}{3x^2} = 4x^3 - x^{-3} + 2x^2 + \frac{2}{3}x^{-2}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 + 3x^{-4} + 4x - \frac{4}{3}x^{-3}$

$= 12x^2 + \frac{3}{x^4} + 4x - \frac{4}{3x^3}$

d $y = \frac{2 - 3x^2 + 5x^4}{x} = 2x^{-1} - 3x + 5x^3$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x^{-2} - 3 + 15x^2 = -\frac{2}{x^2} - 3 + 15x^2$

2 $y = 2(3x^2 - 2x) = 6x^2 - 4x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 12x - 4$

At $(1, -2) \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 8$

Equation of tangent: $y - 2 = 8(x - 1)$
$y = 8x - 6$

3 $y = \frac{x-3}{x} = 1 - 3x^{-1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^{-2} = \frac{3}{x^2}$

At $(-1, 4) \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 3 \quad \therefore$ gradient of normal $= -\frac{1}{3}$

Equation of normal: $y - 4 = -\frac{1}{3}(x + 1)$
$y = -\frac{1}{3}x + \frac{11}{3}$

Exercise 4I

1 **a** $y = (2x + 3)^5 \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 5(2x + 3)^4 (2) = 10(2x + 3)^4$

b $y = \sqrt{2 - 3x} = (2 - 3x)^{\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(2 - 3x)^{\frac{1}{2}}(-3) = \frac{-3}{2\sqrt{2 - 3x}}$

c $y = \frac{2}{x} - 3x + 5x^3$, so $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{x^2} - 3 + 15x^2$

d $y = \frac{-3}{\sqrt{5x^2 + 1}} = -3(5x^2 + 1)^{-\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}(5x^2 + 1)^{-\frac{3}{2}}(10x) = \frac{15x}{\sqrt{(5x^2 + 1)^3}}$

2 $y = \sqrt{3x^2 - 2x} = (3x^2 - 2x)^{\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(3x^2 - 2x)^{-\frac{1}{2}}(6x - 2) = \frac{3x - 1}{\sqrt{3x^2 - 2x}}$

At $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ and $y = 1$, so tangent is

$y - 1 = 2(x - 1)$ i.e. $y = 2x - 1$

3 $y = \frac{x-3}{x} = 1 - \frac{3}{x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{x^2}$

At $(1, -2)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$ so gradient of normal $= -\frac{1}{3}$

$\therefore$ equation of normal is $y + 2 = -\frac{1}{3}(x - 1)$

i.e. $y = -\frac{1}{3}x - \frac{5}{3}$

4 $y=\frac{1}{3x^2-6x+1}=(3x^2-6x+1)^{-1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-(3x^2-6x+1)^{-2}(6x-6)=-\frac{(6x-6)}{(3x^2-6x+1)^2}$$

$6x-6=0 \quad \therefore x=1 \quad y=\frac{1}{3-6+1}=-\frac{1}{2} \quad \left(1,-\frac{1}{2}\right)$

5 $y=\sqrt{1-\sqrt{x}}=\left(1-x^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(1-x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(-\frac{1}{2}x^{-\frac{1}{2}}\right)=-\frac{1}{4\sqrt{1-\sqrt{x}}\sqrt{x}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-1}{4\sqrt{\sqrt{x}-x}}$$

Exercise 4J

1 $y=(x-1)(x+3)^3$

$u(x)=x-1 \qquad u'(x)=1$

$v(x)=(x+3)^3 \qquad v'(x)=3(x+3)^2$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(x-1)3(x+3)^2+(x+3)^3(1)$$
$$=(x+3)^2(3(x-1)+(x+3))$$
$$=(x+3)^2(4x)$$
$$=4x(x+3)^2$$

2 $y=(2x-3)^2(4x+1)^3$

$u(x)=(2x-3)^2 \qquad u'(x)=4(2x-3)$

$v(x)=(4x+1)^3 \qquad v'(x)=12(4x+1)^2$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(2x-3)^2\,12(4x+1)^2+(4x+1)^3\,4(2x-3)$$
$$=4(2x-3)(4x+1)^2[3(2x-3)+(4x+1)]$$
$$=4(2x-3)(4x+1)^2(10x-8)$$
$$=8(2x-3)(4x+1)^2(5x-4)$$

3 $y=\frac{x+1}{x-1}=(x+1)(x-1)^{-1}$

$u(x)=x+1 \qquad u'(x)=1$

$v(x)=(x-1)^{-1} \qquad v'(x)=-(x-1)^{-2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-(x+1)(x-1)^{-2}+(x-1)^{-1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(x-1)^{-2}[-(x+1)+(x-1)]=\frac{-2}{(x-1)^2}$$

4 $y=x\sqrt{1-2x}$

$u(x)=x \qquad u'(x)=1$

$v(x)=(1-2x)^{\frac{1}{2}} \quad v'(x)=\frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2)=-(1-2x)^{-\frac{1}{2}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-x(1-2x)^{-\frac{1}{2}}+(1-2x)^{\frac{1}{2}}$$
$$=(1-2x)^{-\frac{1}{2}}[-x+(1-2x)]=\frac{1-3x}{\sqrt{1-2x}}$$

5 $y=(x^4-3x+1)^{-1}$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-1}{\left(x^4-3x+1\right)^2}(4x^3-3)=\frac{3-4x^3}{\left(x^4-3x+1\right)^2}$$

6 $y=(x-1)^4(3x-2)^{\frac{2}{3}},$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=(x-1)^4\,\frac{2}{3}(3x-2)^{-\frac{1}{3}}\times 3+4(x-1)^3(3x-2)^{\frac{2}{3}}$$
$$=\frac{2(x-1)^4}{(3x-2)^{\frac{1}{3}}}+4(x-1)^3(3x-2)^{\frac{2}{3}}$$
$$=\frac{2(x-1)^3(7x-5)}{(3x-2)^{\frac{1}{3}}}$$

Exercise 4K

1 **a** $y=\frac{x^2-7}{x^3}$

$u(x)=x^2-7 \qquad u'(x)=2x$

$v(x)=x^3 \qquad v'(x)=3x^2$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^3(2x)-(x^2-7)3x^2}{x^6}$$
$$=\frac{2x^2-3x^2+21}{x^4}$$
$$=\frac{21-x^2}{x^4}$$

b $y=\frac{x}{\sqrt{x^2+1}}$

$u(x)=x \qquad u'(x)=1$

$v(x)=(x^2+1)^{\frac{1}{2}} \qquad v'(x)=(x^2+1)^{-\frac{1}{2}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\left((x^2+1)^{\frac{1}{2}}-x^2(x^2+1)^{-\frac{1}{2}}\right)\div(x^2+1)$$
$$=\frac{(x^2+1)-x^2}{(x^2+1)^{\frac{3}{2}}}=\frac{1}{(x^2+1)^{\frac{3}{2}}}$$

c $y=\frac{1}{x^4-3x+1}=(x^4-3x+1)^{-1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-(x^4-3x+1)^{-2}(4x^3-3)$$

$$y=\frac{3-4x^3}{(x^4-3x+1)^2}$$

d $y=\frac{1+\sqrt{x}}{1-\sqrt{x}}$

$u(x)=1+x^{\frac{1}{2}} \qquad u'(x)=\frac{1}{2}x^{-\frac{1}{2}}$

$v(x)=1-x^{\frac{1}{2}} \qquad v'(x)=-\frac{1}{2}x^{-\frac{1}{2}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\left(1-x^{\frac{1}{2}}\right)\frac{1}{2}x^{\frac{1}{2}}+\left(1-x^{\frac{1}{2}}\right)\frac{1}{2}x^{-\frac{1}{2}}}{\left(1-x^{\frac{1}{2}}\right)^2}$$
$$=\frac{x^{-\frac{1}{2}}}{\left(1-x^{\frac{1}{2}}\right)^2}=\frac{1}{\sqrt{x}(1-\sqrt{x})^2}$$

e $y=(\sqrt{x}-x)^{\frac{1}{2}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}(\sqrt{x}-x)^{-\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}-1\right)$$

$$= \frac{1}{4}\left(\sqrt{x} - x\right)^{-\frac{1}{2}}\left(x^{-\frac{1}{2}} - 2\right)$$

$$= \frac{x^{-\frac{1}{2}} - 2}{4\left(\sqrt{x} - x\right)^{\frac{1}{2}}} = \frac{1 - 2\sqrt{x}}{4\sqrt{x}\sqrt{\left(\sqrt{x} - x\right)}} = \frac{1 - 2\sqrt{x}}{4\sqrt{x - x\sqrt{x}}}$$

f $y = \left(\frac{x}{1 - \sqrt{x}}\right)^3 = \frac{x^3}{\left(1 - \sqrt{x}\right)^3}$

$u(x) = x^3$ $\quad u'(x) = 3x^2$

$v(x) = \left(1 - \sqrt{x}\right)^3$ $\quad v'(x) = 3\left(1 - \sqrt{x}\right)^2\left(\frac{-1}{2}x^{\frac{-1}{2}}\right)$

$$\frac{dy}{dx} = \frac{3x^2\left(1 - \sqrt{x}\right)^3 + \frac{3}{2}x^{\frac{5}{2}}\left(1 - \sqrt{x}\right)^2}{\left(1 - \sqrt{x}\right)^6}$$

$$= \frac{3x^2\left(1 - \sqrt{x}\right) + \frac{3}{2}x^{\frac{5}{2}}}{\left(1 - \sqrt{x}\right)^4}$$

$$= \frac{6x^2 - 6x^{\frac{5}{2}} + 3x^{\frac{5}{2}}}{2\left(1 - \sqrt{x}\right)^4}$$

$$= \frac{6x^2 + 3x^{\frac{5}{2}}}{2\left(1 - \sqrt{x}\right)^4}$$

$$= \frac{3x^2\left(2 - \sqrt{x}\right)}{2\left(1 - \sqrt{x}\right)^4}$$

2 $y = \frac{4x}{x^2 + 1}$ $\quad (x = -1)$

$u(x) = 4x$ $\quad u'(x) = 4$

$v(x) = x^2 + 1$ $\quad v'(x) = 2x$

$$\frac{dy}{dx} = \frac{4(x^2 + 1) - 8x^2}{(x^2 + 1)^2} = \frac{4 - 4x^2}{(x^2 + 1)^2}$$

At $x = -1$, gradient $= 0$

3 $y = \frac{8}{4 + x^2} = 8(4 + x^2)^{-1}$ $\quad (x = 1)$

$$\frac{dy}{dx} = -8(4 + x^2)^{-2}(2x) = \frac{-16x}{\left(4 + x^2\right)^2}$$

At $\left(1, \frac{8}{5}\right)$, $\frac{dy}{dx} = \frac{-16}{25}$ $\quad$ gradient of normal $= \frac{25}{16}$

Equation of normal: $y - \frac{8}{5} = \frac{25}{16}(x - 1)$, $y = \frac{25}{16}x + \frac{3}{80}$

4 $f(x) = \sqrt[3]{\left(1 - \frac{1}{2 + x}\right)^2} = \left(1 - (2 + x)^{-1}\right)^{\frac{2}{3}}$

$$f'(x) = \frac{2}{3}\left(1 - (2 + x)^{-1}\right)^{-\frac{1}{3}}(2 + x)^{-2}$$

$$= \frac{2}{3(2 + x)^2\sqrt[3]{1 - \frac{1}{2 + x}}}$$

Exercise 4L

1 $f(x) = 4x + 1 + \frac{1}{x}$

$f'(x) = 4 - x^{-2}$ $\quad f''(x) = 2x^{-3} = \frac{2}{x^3}$

2 $f(x) = x^4 - 2x - 1$

$f'(x) = 4x^3 - 2$ $\quad f''(x) = 12x^2$

$f'(0) = -2$ $\quad f''(-1) = 12$

3 $f(x) = x^4 - 4x^3 + 16x - 16$

$f'(x) = 4x^3 - 12x^2 + 16$

$f''(x) = 12x^2 - 24x$

$f(x) = (x + 2)(x - 2)^3 \Rightarrow x = -2$ or $x = 2$

$f(-2) \neq f'(-2) \neq f''(-2)$,
but $f(2) = f'(2) = f''(2) = 0$ so $x = 2$

4 $f(x) = x^4 + rx^2 + sx + t$

$f(-1) = 16 \Rightarrow r - s + t = 15$ $\quad (1)$

$f'(x) = 4x^3 + 2rx + s$

$f'(-1) = -16 \Rightarrow s - 2r = -12$ $\quad (2)$

$f''(x) = 12x^2 + 2r$

$f''(-1) = 16 \Rightarrow 12 + 2r = 16$ $\quad (3)$

Solve equations (1), (2), (3) to find $r = 2$, $s = -8$, $t = 5$.

5 $s(t) = (t - 4)^3(3 - 2t)^2$

a Velocity $= s'(t) = (t - 4)^3 \, 2(3 - 2t)(-2)$

$= 3(t - 4)^2(3 - 2t)^2$

$s'(t) = (t - 4)^2(3 - 2t)[-4(t - 4) + 3(3 - 2t)]$

$= (t - 4)^2(3 - 2t)(25 - 10t)$

$s'(4) = 0$ ms^{-1}

b $s'(t) = (t - 4)^2(75 - 80t + 20t^2)$

$s''(t) = (t - 4)^2(-80 + 40t) + 2(t - 4)$
$(70 - 80t + 20t^2)$

$= (t - 4)[-80t + 40t^2 + 320 - 160t + 150$
$- 160t + 40t^2]$

$= (t - 4)(80t^2 - 400t + 470)$

acceleration $= s''(4) = 0$ ms^{-2}

c $s''(t) = 80t^3 - 720t^2 + 2070t - 1880$

jerk $= s'''(t) = 240t^2 - 1440t + 2070$

$s'''(1) = 240 - 1440 + 2070 = 870$ ms^{-1}

6 $f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}$ $\quad f''(x) = 2x^{-3}$ $\quad f'''(x) = -6x^{-4}$

$f^{(4)}(x) = 24x^{-5}$ $\quad f^{(5)}(x) = -120x^{-6}$

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$$

Exercise 4M

1 **a** $y = x^2 - 3x + 1$

$\frac{dy}{dx} = 2x - 3$

$2x - 3 = 0 \quad \therefore\ x = \frac{3}{2} \qquad y = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 1 = -\frac{5}{4}$

x	$x < \frac{3}{2}$	$x > \frac{3}{2}$
$\frac{dy}{dx}$	–	+

$\therefore$ minimum value $= \frac{-5}{4} \quad \left(\text{when } x = \frac{3}{2}\right)$

b $y = -2x^3 + 6x^2 - 3)$

$\frac{dy}{dx} = -6x^2 + 12x$

$-6x^2 + 12x = 0 \quad \therefore\ 6x(-x + 2) = 0$

$x = 0 \quad$ or 2 $\quad (0, -3) \quad (2, 5)$

x	$x < 0$	$0 < x < 2$	$x > 2$
$\frac{dy}{dx}$	–	+	–

$\therefore$ minimum value $= -3 \quad$ (at $x = 0$)

maximum value $= 5 \quad$ (at $x = 2$)

c $y = 3x^4 - 2x^3 - 3x^2 + 4$

$\frac{dy}{dx} = 12x^3 - 6x^2 - 6x$

$12x^3 - 6x^2 - 6x = 0 \quad 6x(2x^2 - x - 1) = 0$

$6x(2x + 1)(x - 1) = 0$

$x = 0, \ \frac{-1}{2}$ or 1 $\quad (0, 4) \quad \left(\frac{-1}{2}, \frac{59}{16}\right) \quad (1, 2)$

x	$x < \frac{-1}{2}$	$\frac{-1}{2} < x < 0$	$0 < x < 1$	$x > 1$
$\frac{dy}{dx}$	–	+	–	+

$\therefore$ minimum values are $\frac{59}{16}$ at $x = -\frac{1}{2}$ and

2 (at $x = 1$) maximum value $= 4$ (at $x = 0$)

d $y = x^4 - 4x^3$

$\frac{dy}{dx} = 4x^3 - 12x^2$

$4x^3 - 12x^2 = 0 \qquad 4x^2(x - 3) = 0$

$x = 0$ or 3 $\qquad (0, 0) \qquad (3, -27)$

x	$x < 0$	$0 < x < 3$	$x > 3$
$\frac{dy}{dx}$	–	–	+

$\therefore$ horizontal point of inflexion at $(0, 0)$

minimum value $= -27$ (at $x = 3$)

Exercise 4N

1 **a** **i** $x = -1$ or 1 $\qquad$ **ii** $]-\infty, -1\,[\cup]\ 1, \infty[$

iii $]-1, 1[$

b **i** $x = -1, \frac{1}{2}, \frac{3}{2}$ $\qquad$ **ii** $]-1, \frac{1}{2}[\ \cup\ ]\frac{3}{2}, \infty[$

iii $]-\infty, -1[\ \cup\ ]\frac{1}{2}, \frac{3}{2}[$

2 **a** **i** $x = -\frac{1}{2}$ $\qquad$ **ii** $]-\frac{1}{2}, \infty[$

iii $]-\infty, -\frac{1}{2}[$

3 **a** $y = -3x^2 + 6x - 1$

$\frac{dy}{dx} = -6x + 6 \quad -6x + 6 = 0 \quad \Rightarrow x = 1 \quad (1, 2)$

x	$x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	+	0	–

maximum at $(1, 2)$

f is increasing for $-\infty, 1[$

f is decreasing for $1, \infty[$

b $y = x\sqrt{2 - x^2} \qquad \left(-\sqrt{2} \le x \le \sqrt{2}\right)$

$\frac{dy}{dx} = x\,\frac{1}{2}(2 - x^2)^{-\frac{1}{2}}(-2x) + (2 - x^2)^{\frac{1}{2}}$

$= \frac{-x^2}{\left(2 - x^2\right)^{\frac{1}{2}}} + \left(2 - x^2\right)^{\frac{1}{2}}$

$= \frac{-x^2 + (2 - x^2)}{\sqrt{2 - x^2}} = \frac{2 - 2x^2}{\sqrt{2 - x^2}}$

$\frac{2 - 2x^2}{\sqrt{2 - x^2}} = 0 \quad \Rightarrow x = \pm 1\ (1, 1) \quad (-1, -1)$

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	–	0	+	0	–

minimum at $(-1, -1)$, maximum at $(1, 1)$

increasing for $]-1, 1[$

decreasing for $\left[-\sqrt{2}, -1\right[\ \cup\ \left]1, \sqrt{2}\right]$

c $y = \frac{x}{x^2 + 1} \quad \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$

$\frac{dy}{dx} = 0$ when $x = \pm 1 \quad \left(1, \frac{1}{2}\right) \quad \left(-1, \frac{-1}{2}\right)$

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	–	0	+	0	–

minimum at $\left(-1, -\frac{1}{2}\right)$, maximum at $\left(1, \frac{1}{2}\right)$

increasing for $]-1, 1[$

decreasing for $]-\infty, -1\,[\cup]\ 1, \infty[$

d $y = x^{\frac{1}{3}}(x - 2) = x^{\frac{4}{3}} - 2x^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{4}{3}x^{\frac{1}{3}} - \frac{2}{3}x^{-\frac{2}{3}}, \quad \frac{4}{3}x^{\frac{1}{3}} - \frac{2}{3}x^{-\frac{2}{3}} = 0$

$\therefore 4x - 2 = 0 \quad \therefore x = \frac{1}{2} \quad \left(\frac{1}{2}, \frac{-3}{2(2)^{\frac{1}{3}}}\right)$

x	**$x < \frac{1}{2}$**	**$x = \frac{1}{2}$**	**$x > \frac{1}{2}$**
$\frac{dy}{dx}$	–	0	+

minimum at $\left(\frac{1}{2}, \frac{3}{2^{\frac{4}{3}}}\right)$

increasing for $\left]\frac{1}{2}, \infty\right[$, decreasing for $\left]-\infty, \frac{1}{2}\right[$

e $y = x^2\sqrt{2 - x^2} \qquad \left(-\sqrt{2} \le x \le \sqrt{2}\right)$

$\frac{dy}{dx} = x^2\,\frac{1}{2}\,(2 - x^2)^{-\frac{1}{2}}(-2x) + 2x(2 - x^2)^{\frac{1}{2}}$

$= \frac{-x^3 + 2x(2 - x^2)}{\left(2 - x^2\right)^{\frac{1}{2}}} = \frac{4x - 3x^3}{\sqrt{2 - x^2}}$

$\frac{dy}{dx} = 0 \Rightarrow 4x - 3x^3 = 0, \; x(4 - 3x^2) = 0$

$x = 0, \pm\frac{2}{\sqrt{3}} \; (0, 0), \left(\frac{2}{\sqrt{3}}, \frac{4}{3}\sqrt{\frac{2}{3}}\right), \left(-\frac{2}{\sqrt{3}}, \frac{4}{3}\sqrt{\frac{2}{3}}\right)$

x	**$x < -\frac{2}{\sqrt{3}}$**	**$x = -\frac{2}{\sqrt{3}}$**	**$-\frac{2}{\sqrt{3}} < x < 0$**	**$x = 0$**
$\frac{dy}{dx}$	+	0	–	0

x	**$0 < x < \frac{2}{\sqrt{3}}$**	**$x = \frac{2}{\sqrt{3}}$**	**$x > \frac{2}{\sqrt{3}}$**
$\frac{dy}{dx}$	+	0	–

maxima at $\left(\frac{-2}{\sqrt{3}}, \frac{4}{3}\sqrt{\frac{2}{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \frac{4}{3}\sqrt{\frac{2}{3}}\right)$,

minimum at $(0, 0)$,

increasing for $\left]-\sqrt{2}, -\frac{2}{\sqrt{3}}\right[\; \cup \; \left]0, \frac{2}{\sqrt{3}}\right[$

decreasing for $\left]-\frac{2}{\sqrt{3}}, 0\right[\; \cup \; \left]\frac{2}{\sqrt{3}}, \sqrt{2}\;\right]$

Exercise 40

1 **a** $y = 2x^3 + 3x^2 - 12x - 3$

$\frac{dy}{dx} = 6x^2 + 6x - 12$

$6(x^2 + x - 2) = 0$

$6(x + 2)(x - 1) = 0$

$x = -2$ or 1

$\frac{d^2y}{dx^2} = 12x + 6$

$f''(-2) = -18 < 0 \quad \therefore f$ has a maximum at $(-2, 17)$

$f''(1) = 18 > 0 \quad \therefore f$ has a minimum at $(1, -10)$

b $y = -x^4 + 2x - 1$

$\frac{dy}{dx} = -4x^3 + 2 \quad -4x^3 + 2 = 0 \; \therefore x^3 = \frac{1}{2} \;\; x = \frac{1}{\sqrt[3]{2}}$

$= 0.794$

$\frac{d^2y}{dx^2} = -12x^2 < 0 \; \therefore$ maximum at $(0.794, 0.191)$

c $y = x^5 - 5x$

$\frac{dy}{dx} = 5x^4 - 5 \quad 5x^4 - 5 = 0, \quad x^4 = 1, \quad x = \pm 1$

$\frac{d^2y}{dx^2} = 20x^3$

$f''(-1) < 0 \quad \therefore$ maximum at $(-1, 4)$

$f''(1) > 0 \quad \therefore$ minimum at $(1, -4)$

d $y = \frac{12}{x^2 + 2x - 3} = 12(x^2 + 2x - 3)^{-1}$

$\frac{dy}{dx} = -12(x^2 + 2x - 3)^{-2}(2x + 2) = \frac{-24(x + 1)}{\left(x^2 + 2x - 3\right)^2}$

$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = -1$

x	**$x < -1$**	**$x = -1$**	**$x > -1$**
$\frac{dy}{dx}$	+	0	–

maximum at $(-1, -3)$

e $y = \frac{3x + 3}{x(3 - x)} = \frac{3x + 3}{3x - x^2}$

$\frac{dy}{dx} = \frac{(3x - x^2)3 - (3x + 3)(3 - 2x)}{\left(3x - x^2\right)^2}$

$= \frac{9x - 3x^2 - 9x + 6x^2 - 9 + 6x}{\left(3x - x^2\right)^2}$

$= \frac{3x^2 + 6x - 9}{x^2(3 - x)^2} = \frac{3(x^2 + 2x - 3)}{x^2(3 - x)^2} = \frac{3(x + 3)(x - 1)}{x^2(3 - x)^2}$

$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = -3$ or 1

x	**$x < -3$**	**$x = -3$**	**$-3 < x < 1$**	**$x = 1$**	**$x > 1$**
$\frac{dy}{dx}$	+	0	–	0	+

maximum at $\left(-3, \frac{1}{3}\right)$, minimum at $(1, 3)$

2 **b** $y = \frac{1 - x}{x^2 + 8}$

i $\frac{dy}{dx^2} = \frac{(-x^2 + 8) - 2x(1 - x)}{(x^2 + 8)^2} = \frac{x^2 - 2x - 8}{(x^2 + 8)^2}$

$\frac{dy}{dx} = 0 \Rightarrow (x - 4)(x + 2) = 0$

$x = -2$ or 4

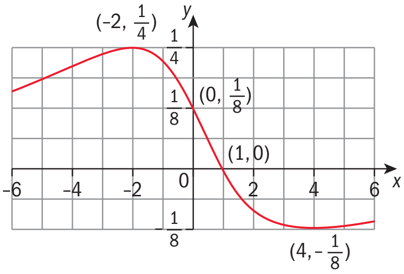
x	**$x < -2$**	**$x = -2$**	**$-2 < x < 4$**	**$x = 4$**	**$x > 4$**
$\frac{dy}{dx}$	+	0	–	0	+

maximum at $\left(-2, \frac{1}{4}\right)$ minimum at $\left(4, \frac{-1}{8}\right)$

ii f is increasing for $]-\infty, -2[\cup]4, \infty[$

f is decreasing for $]-2, 4[$

iii



Exercise 4P

1 **a** $y = x^3 - x$

i $\frac{dy}{dx} = 3x^2 - 1 \qquad \frac{d^2y}{dx^2} = 6x$

$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad x = 0$

x	$x < 0$	$x = 0$	$x > 0$
$\frac{d^2y}{dx^2}$	–	0	+

point of inflexion at $(0, 0)$

ii concave up for $]0, \infty[$

concave down for $]-\infty, 0[$

b $y = x^4 - 3x + 2$

i $\frac{dy}{dx} = 4x^3 - 3 \qquad \frac{d^2y}{dx^2} = 12x^2$

$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad x = 0$

x	$x < 0$	$x = 0$	$x > 0$
$\frac{d^2y}{dx^2}$	+	0	+

no point of inflexion

ii concave up for $]-\infty, 0[\cup]0, \infty[$

c $y = \sqrt{4x - x^2}$ where $x \in [0, 4]$

i $\frac{dy}{dx} = \frac{1}{2}\left(4x - x^2\right)^{-\frac{1}{2}}(4 - 2x) = \frac{2 - x}{\left(4x - x^2\right)^{\frac{1}{2}}}$

$$\frac{d^2y}{dx^2} = \frac{-\left(4x - x^2\right)^{\frac{1}{2}} - (2 - x)\frac{1}{2}\left(4x - x^2\right)^{-\frac{1}{2}}\left(4 - 2x\right)}{\left(4x - x^2\right)}$$

$$= \frac{-\left(4x - x^2\right) - (2 - x)^2}{\left(4x - x^2\right)^{\frac{3}{2}}}$$

$$= \frac{-4x + x^2 - 4 + 4x - x^2}{\left(4x - x^2\right)^{\frac{3}{2}}}$$

$$= \frac{-4x}{\left(4x - x^2\right)^{\frac{3}{2}}}$$

$\frac{d^2y}{dx^2} < 0 \quad \therefore$ no points of inflexion

ii concave down for $[0, 4]$

d $y = (x - 1)^{\frac{2}{3}}$

i $\frac{dy}{dx} = \frac{2}{3}(x - 1)^{-\frac{1}{3}} \qquad \frac{d^2y}{dx^2} = -\frac{2}{9}(x - 1)^{-\frac{4}{3}}$

$\frac{d^2y}{dx^2} = \frac{-2}{9(x - 1)^{\frac{4}{3}}} \qquad \frac{d^2y}{dx^2} \neq 0$

$\therefore$ no points of inflexion

ii

x	$x < 1$	$x > 1$
$\frac{d^2y}{dx^2}$	–	0

concave down for $]-\infty, 1[\cup]1, \infty[$

e $y = \frac{3x^2}{x - 1}$

i $\frac{dy}{dx} = \frac{6x(x - 1) - 3x^2}{(x - 1)^2} = \frac{3x^2 - 6x}{(x - 1)^2}$

$$\frac{d^2y}{dx^2} = \frac{(x - 1)^2(6x - 6) - (3x^2 - 6x)2(x - 1)}{(x - 1)^4}$$

$$= \frac{(x - 1)(6x - 6) - (3x^2 - 6x)2}{(x - 1)^3}$$

$$= \frac{6x^2 - 12x + 6 - 6x^2 + 12x}{(x - 1)^3}$$

$$= \frac{6}{(x - 1)^3}$$

$\frac{d^2y}{dx^2} \neq 0 \quad \therefore$ no points of inflexion

ii

x	$x < 1$	$x > 1$
$\frac{d^2y}{dx^2}$	–	+

concave up for $]1, \infty[$, concave down for $]-\infty, 1[$

Exercise 4Q

1 $s(t) = -5t^2 + 5t + 10$

a $s(0) = 10$ m

b $-5t^2 + 5t + 10 = 0$

$t^2 - t - 2 = 0$

$(t - 2)(t + 1) = 0$

$\therefore t = 2$ seconds

c $v(t) = -10t + 5 \qquad a(t) = -10$

$v(2) = -15\,\text{ms}^{-1} \qquad a(2) = -10\,\text{ms}^{-2}$

The diver is moving downwards and speeding up when he hits the water.

2 $s = 50t - 15t^2$

a $v = 50 - 30t = 0$ when $t = \frac{5}{3}$

maximum height $= 5\left(\frac{5}{3}\right) = 41\frac{2}{3}$ m

b $20 = 50t - 15t^2$

$3t^2 - 10t + 4 = 0$

$t = 0.4648$ or 2.8685

$v = 50 - 30t \quad v(0.4648) = 36.1 \text{ ms}^{-1}$

$v(2.8685) = -36.1 \text{ ms}^{-1}$

speed $= 36.1 \text{ ms}^{-1}$ upwards (when $t = 0.4648$) and downwards (when $t = 2.8685$)

c $a = -30 \text{ ms}^{-2}$

d $50t - 15t^2 = 0$

$5t(10 - 3t) = 0$

$\therefore$ rock hits the ground again when $t = \frac{10}{3}$ s

3 $s = 7t + 5t^2 - 2t^3$

a $v = 7 + 10t - 6t^2 \quad a = 10 - 12t$

$v(0) = 7 \text{ ms}^{-1} \quad a(0) = -10 \text{ ms}^{-2}$

Initially the particle is moving in a positive direction and is slowing down.

b $v(2) = 3 \text{ ms}^{-1} \quad a(2) = -14 \text{ ms}^{-2}$

The particle is moving in a positive direction and slowing down.

4 $s = 10t^2 - t^3$

a $s(3) = 63\text{m} \quad \therefore$ average velocity $= \frac{63}{3} = 21 \text{ ms}^{-1}$

b $v = 20t - 3t^2 \quad a = 20 - 6t$

$v(3) = 33 \text{ ms}^{-1} \quad a(3) = 2 \text{ ms}^{-2}$

c Speeding up.

d $20t - 3t^2 = 0$

$t(20 - 3t) = 0$

$t = 0$ or $\frac{20}{3}$ direction changes when $t = \frac{20}{3}$ s

5 $s(t) = \frac{1}{3}t^3 - 3t^2 + 8t$

a $v(t) = t^2 - 6t + 8 \quad a(t) = 2t - 6$

b **i** $t^2 - 6t + 8 = 0$

$(t - 2)(t - 4) = 0$

$t = 2$ s or 4 s

ii

t	**0 < t < 2**	**2 < t < 4**	**t > 4**
v	+	−	+
t	0 < t < 3	t > 3	
a	−	+	

v and a have the same sign for $2 < t < 3$ and $t > 4$ $\therefore$ the particle is speeding up at these times.

iii the particle is slowing down for $0 < t < 2$ and $3 < t < 4$

c $a(2) = -2 \text{ ms}^{-1} \quad a(4) = 2 \text{ ms}^{-2}$

the particle changes direction from positive to negative when $t = 2s$ and from negative to positive when $t = 4s$.

d $t = 2s$ and $t = 4s$.

e $0 - 2s$ distance $= s(2) - s(0) = 6\frac{2}{3} - 0 = 6\frac{2}{3}$

$2 - 4s$ distance $= s(4) - s(2) = 5\frac{1}{3} - 6\frac{2}{3} = -1\frac{1}{3}$

$4 - 5s$ distance $= s(5) - s(4) = 6\frac{2}{3} - 5\frac{1}{3} = 1\frac{1}{3}$

total distance $= 6\frac{2}{3} + 1\frac{1}{3} + 1\frac{1}{3} = 9\frac{1}{3}$ m

Exercise 4R

1 $c(x) = 20\,000 + 180x - 0.1x^2$

a $c'(x) = 180 - 0.2x$

b $c'(100) = 180 - 0.2 \times 100$

$= 160$ euros / tank

c $c(101) - c(100) = 159.9 \Rightarrow$ cost of producing 1 extra tank is nearly the same as the marginal cost function.

2 **a** **i** $p(x)$ must be > 0 so $0.002x < 7$ i.e. $x < 3500$

$\therefore$ domain is $0 < x < 3500$

ii $c'(x) = 3$ euros / unit $\Rightarrow$ it will always cost 3 euros to make an extra memory stick

iii $r(x) = x(7 - 0.002x)$

b Break-even points: $r(x) = c(x)$ when $7x = 0.002x^2 = 500 + 3x$ i.e. $0.002x^2 - 4x + 500 = 0$

$\Rightarrow x = \frac{4 \pm \sqrt{16 - 4}}{0.004}$

$= 134$ or 1870 (3 sf).

For profit, need to make x memory sticks where $134 < x < 1870$.

3 Average cost per 100 units $= \frac{c(x)}{x} = 500 + \frac{1000}{x}$

To minimize cost, need $\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0$

$\Rightarrow 500 - \frac{1000}{x^2} = 0$

$\Rightarrow x^2 = 2 \Rightarrow x = 1.41$ (3 sf)

$\therefore$ costs are minimised by making 141 units.

4 **b** $r(x) = 35x - 3$ and $p(x) = r(x) - c(x)$

so $p(x) = 35x - 3 - 400 - 20x + 0.2x^2 - 0.0004x^3$

$= 15x - 403 + 0.2x^2 - 0.0004x^3$

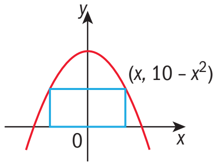
$p'(x) = 15 + 0.4x - 0.0012x^2 = 0$

$\Rightarrow x = 367$, so 367 jackets must be made to maximise profit.

c Minimising costs will not necessarily maximise profits.

Exercise 4S

1



$A = 2x(10 - x^2)$

$\quad = 20x - 2x^3$

$A' = 20 - 6x^2$

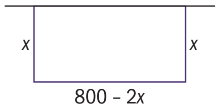
$20 - 6x^2 = 0 \Rightarrow x = \sqrt{\frac{10}{3}}$

$A'' = -12x$

If $x = \sqrt{\frac{10}{3}}$, $\quad A'' < 0 \quad \therefore$ max.

base = $2\sqrt{\frac{10}{3}}$ $\quad$ height = $10 - \frac{10}{3} = \frac{20}{3}$ (3.65 by 6.67)

2



$A = x(800 - 2x)$

$\quad = 800x - 2x^2$

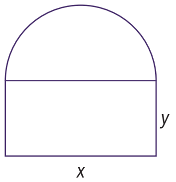
$A' = 800 - 4x$

$A' = 0 \quad \Rightarrow \quad x = 200$

$A'' = -4 < 0 \quad \therefore$ maximum

maximum area = $200 \times 400 = 80\,000 \text{ m}^2$

3



$x + 2y + \frac{\pi x}{2} = 4$

$2x + 4y + \pi x = 8$

$y = \frac{8 - 2x - \pi x}{4}$

$A = xy = \frac{1}{4}(8x - 2x^2 - \pi x^2)$

$A' = \frac{1}{4}(8 - 4x - 2\pi x)$

$A = 0 \Rightarrow x(4 + 2\pi)\ 8$

$x = \frac{8}{4 + 2\pi} = 0.778 \quad y = 1$

$A'' = \frac{1}{4}(-4 - 2\pi) < 0 \quad \therefore$ maximum

dimensions: $\frac{8}{4 + 2\pi}$ m by 1m

Exercise 4T

1 **a** $3y^2 + x^2 = 4$

$6y\frac{dy}{dx} + 2x = 0 \quad \therefore \frac{dy}{dx} = \frac{-x}{3y}$

b $y^4 = x^3 + 1$

$4y^3\frac{dy}{dx} = 3x^2 \quad \therefore \frac{dy}{dx} = \frac{3x^2}{4y^3}$

c $x^2 + y^2 - 3x + 4y = 2$

$2x + 2y\frac{dy}{dx} - 3 + 4\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{3 - 2x}{2y + 4}$

d $2x^2 - 3x^2y^2 + y^2 = 9$

$4x - 3(x^2\, 2y\frac{dy}{dx} + 2xy^2) + 2y\frac{dy}{dx} = 0$

$4x - 6x^2y\frac{dy}{dx} - 6xy^2 + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{3xy^2 - 2x}{y - 3x^2y}$

e $(x + y)^2 = 5 - 2x$

$2(x + y)\left(1 + \frac{dy}{dx}\right) = -2$

$1 + \frac{dy}{dx} = -\frac{1}{x + y}$

$\frac{dy}{dx} = -\frac{1}{x + y} - 1 = -\frac{(1 + x + y)}{x + y}$

f $x^2 = \frac{x - y}{x + y} \quad x^3 + x^2y = x - y$

$3x^2 + x^2\frac{dy}{dx} + 2xy = 1 - \frac{dy}{dx}$

$\frac{dy}{dx}(x^2 + 1) = 1 - 3x^2 - 2xy$

$\frac{dy}{dx} = \frac{1 - 3x^2 - 2xy}{x^2 + 1}$

2 $x^2 - y^2 = 9 \qquad (5, 4)$

$2x - 2y\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{x}{y} = \frac{5}{4}$

$y - 4 = \frac{5}{4}(x - 5)$

$y = \frac{5}{4}x - \frac{9}{4}$

3 $y^2 = 3x + 1 \qquad (1, -2)$

$2y\frac{dy}{dx} = 3 \quad \frac{dy}{dx} = \frac{3}{2y} = \frac{-3}{4}$

$y + 2 = \frac{4}{3}(x - 1)$

$y = \frac{4}{3}x - \frac{10}{3}$

4 $x^2 - \sqrt{3}xy + 2y^2 = 5 \qquad (\sqrt{3}, 2)$

$2x - \sqrt{3}\left(x\frac{dy}{dx} + y\right) + 4y\frac{dy}{dx} = 0$

$-\sqrt{3}\frac{dy}{dx} + 4y\frac{dy}{dx} = \sqrt{3}y - 2x$

$\frac{dy}{dx} = \frac{\sqrt{3}y - 2x}{4y - \sqrt{3}x} = 0$

Tangent: $y = 2 \qquad$ Normal: $x = \sqrt{3}$

5 $x^2 + y^2 - 6x - 8y = 0$

$2x + 2y\frac{dy}{dx} - 6 - 8\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{3 - x}{y - 4} \quad \frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 3$

$9 + y^2 - 18 - 8y = 0$

$y^2 - 8y - 9 = 0$

$(y - 9)(y + 1) = 0$

$y = 9$ or -1

stationary points (3, 9), (3, −1)

6 $3x^2 + 2xy + y^2 = 3 \qquad (1, -2)$

$$6x + 2x\frac{dy}{dx} + 2y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(3x + y)}{x + y} = -\frac{(3 - 2)}{-1} = 1$$

$$3x + x\frac{dy}{dx} + y + y\frac{dy}{dx} = 0$$

$$3 + x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$3 + \frac{d^2y}{dx^2} + 2 - 2\frac{d^2y}{dx^2} + 1 = 0$$

$$\frac{d^2y}{dx^2} = 6$$

7 $x^2 + xy + y^2 = 3$

$y = 0 \quad x^2 = 3 \quad x = \pm\sqrt{3}$

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(2x + y)}{x + 2y}$$

At $(\sqrt{3}, 0)$ $\frac{dy}{dx} = -2$ At $(-\sqrt{3}, 0)$ $\frac{dy}{dx} = -2$

∴ tangents are parallel

9 $x + y = x^2 - 2xy + y^2$

a $$1 + \frac{dy}{dx} = 2x - 2\left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx}$$

$$1 + \frac{dy}{dx} = 2x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - 2y - 1$$

$$\frac{dy}{dx} = \frac{2x - 2y - 1}{2x - 2y + 1}$$

b $$1 - \frac{dy}{dx} = 1 - \frac{(2x - 2y - 1)}{(2x - 2y + 1)}$$

$$= \frac{2x - 2y + 1 - 2x + 2y + 1}{2x - 2y + 1}$$

$$1 - \frac{dy}{dx} = \frac{2}{2x - 2y + 1}$$

c $$\frac{d^2y}{dx^2} = \frac{(2x - 2y + 1)\left(2 - 2\frac{dy}{dx}\right) - (2x - 2y - 1)\left(2 - 2\frac{dy}{dx}\right)}{(2x - 2y + 1)^2}$$

$$= \frac{4 - 4\frac{dy}{dx}}{(2x - 2y + 1)^2}$$

$$= \frac{4 - 4\frac{(2x - 2y - 1)}{2x - 2y + 1}}{(2x - 2y + 1)^2}$$

$$= \frac{4(2x - 2y + 1) - 4(2x - 2y - 1)}{(2x - 2y + 1)^3}$$

$$= \frac{8}{(2x - 2y + 1)^3} = \left(\frac{2}{2x - 2y + 1}\right)^3$$

$$\therefore \frac{d^2y}{dx^2} = \left(1 - \frac{dy}{dx}\right)^3$$ (from **b**)

Exercise 4U

1 $A = \pi r^2 \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

2 $A = 2\pi r^2 + 2\pi rh$

$$\frac{dA}{dt} = 4\pi r\frac{dr}{dt} + 2\pi r\frac{dh}{dt} + 2\pi r\frac{dr}{dt}$$

$$\frac{dA}{dt} = (4\pi r + 2\pi h)\frac{dr}{dt} + 2\pi r\frac{dh}{dt}$$

3 Let x = diagonal of the box.

$x^2 = l^2 + w^2 + h^2$

$$2x\frac{dx}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt} + 2h\frac{dh}{dt}$$

$$\frac{dx}{dt} = \frac{l\frac{dl}{dt} + w\frac{dw}{dt} + h\frac{dh}{dt}}{(l^2 + w^2 + h^2)^{\frac{1}{2}}}$$

4 $\frac{dl}{dt} = 2 \text{ cm s}^{-1} \quad \frac{dw}{dt} = -2 \text{ cm s}^{-1}\ l = 12 \text{ cm}, w = 5 \text{ cm}$

a $A = lw$

$$\frac{dA}{dt} = l\frac{dw}{dt} + w\frac{dl}{dt}$$

$= 12(-2) + 5(2)$

$= -14 \text{ cm}^2\text{s}^{-1}$

b $p = 2l + 2w$

$$\frac{dp}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}$$

$= 2(2) + 2(-2)$

$= 0 \text{ cms}^{-1}$

c Let x = diagonal of the rectangle.

$x^2 = l^2 + w^2$

$$2x\frac{dx}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt} \qquad (l = 12, w = 5, x = 13)$$

$$13\frac{dx}{dt} = 12(2) + 5(-2)$$

$$\frac{dx}{dt} = \frac{14}{13}\text{cms}^{-1}$$

5 $\frac{dv}{dt} = 1.5 \text{ m}^3\text{s}^{-1} \quad v = 81 \text{ m}^3 \quad x = \sqrt[3]{81} \text{ m}$

Let x = side length of cube

$V = x^3 \qquad A = 6x^2$

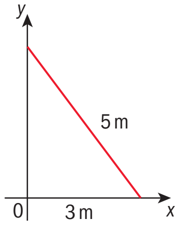
$$\frac{dA}{dt} = \frac{dv}{dt} \times \frac{dx}{dv} \times \frac{dA}{dx}$$

$$= 1.5 \times \frac{1}{3x^2} \times 12x$$

$$= \frac{6}{x}$$

$$\frac{dA}{dt} = \frac{6}{\sqrt[3]{81}} = 1.39 \text{ m}^2\text{s}^{-1}$$

6



When $x = 3$ m, $\frac{dx}{dt} = 0.5 \text{ ms}^{-1}$

a $x^2 + y^2 = 25$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$3(0.5) + 4\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -0.375$$

$\therefore$ 0.375 ms^{-1} down the wall

b $A = \frac{1}{2}xy \quad \frac{dA}{dt} = \frac{1}{2}x\frac{dy}{dt} + \frac{1}{2}y\frac{dx}{dt}$

$$= \frac{1}{2}(3)(-0.375) + \frac{1}{2}(4)0.5$$

$$= 0.4375 \text{ m}^2\text{s}^{-1}$$

7 $\frac{dA}{dt} = 2 \text{ cm}^2\text{s}^{-1}$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} \qquad (r = 5)$

$2 = 10\pi\frac{dr}{dt} \qquad \frac{dr}{dt} = \frac{1}{5\pi} = 0.0637 \text{ cms}^{-1}$

10 $\frac{dr_1}{dt} = 1.2 \text{ ms}^{-1} \qquad \frac{dr_2}{dt} = 1.5 \text{ ms}^{-1}$

$A = \pi r_1^2 - \pi r_2^2$

$$\frac{dA}{dt} = 2\pi r_1\frac{dr_1}{dt} - 2\pi r_2\frac{dr_2}{dt}$$

$$= 2\pi(9 \times 1.2 - 1 \times 1.5)$$

$$= 18.6\pi = 58.4 \text{ m}^2\text{s}^{-1}$$

Review exercise

1 **a** $\lim\limits_{x\to 1}\frac{x^3-3}{x+1} = -1$

b $\lim\limits_{x\to 0}\frac{\sqrt{x^2-1}}{x}$ does not exist since the domain is $]-\infty, -1] \cup [1, \infty[$

c $\lim\limits_{x\to 0}\frac{3^x-1}{x} = 1.10$ (3sf)

d $\lim\limits_{x\to 0}\frac{3x^2+x^2}{x^2} = 4$

e $\lim\limits_{x\to\infty}\frac{5x^2}{2x^3+1} = 0$

f $\lim\limits_{x\to -\infty}\frac{7}{x^3+1} = 0$

2 $y = \begin{cases} x^2 + 2x & x \le 2 \\ x^3 - 6x & x > 2 \end{cases}$

$f(2) = 8 \qquad \lim\limits_{x\to 2^-} f(x) = -4$

$f(x)$ is not continuous at $x = 2$

3 $a_n = \frac{2n^2-3}{n^3-2}$ sequence converges since $a_n \to 0$ as $n \to \infty$.

4 $\sum\limits_{n=0}^{\infty} 3\left(\frac{(-1)^n}{5^n}\right)$ is a geometric series with $r = -\frac{1}{5}$ hence it converges.

$$\sum_{n=0}^{\infty} 3\left(\frac{(-1)^n}{5^n}\right) = \frac{3}{1-\left(-\frac{1}{5}\right)} = 2.5$$

5 Geometric series with, $r = \frac{1}{1+a^2}$

$\frac{1}{1+a^2} < 1$ provided $a \neq 0$

$$\text{sum} = \frac{a^2}{1-\frac{1}{1+a^2}} = \frac{a^2(1+a^2)}{1+a^2-1} = 1+a^2$$

6 $y = \frac{x^3-2x^2+5}{x^2-x^3}$

a $y = -1$

b $\frac{x^3-2x^2+5}{x^2-x^3} = -1$

$$x^3 - 2x^2 + 5 = -x^2 + x^3$$

$$5 = x^2$$

$$x = \pm\sqrt{5}$$

$(\sqrt{5}, -1) \qquad (-\sqrt{5}, -1)$

7 $y = \frac{2x+1}{x^2+1} \qquad (0, 1)$

$$\frac{dy}{dx} = \frac{2(x^2+1) - 2x(2x+1)}{(x^2+1)^2}$$

If $x = 0$, $\frac{dy}{dx} = 2$

Tangent: $y - 1 = 2x \qquad y = 2x + 1$

Normal: $y - 1 = -\frac{1}{2}x \qquad y = -\frac{1}{2}x + 1$

9 $x + y = -3 \qquad$ gradient $= -1$

$y = x\sqrt{x+1}$

$$\frac{dy}{dx} = x\frac{1}{2}(x+1)^{-\frac{1}{2}} + (x+1)^{\frac{1}{2}}$$

$$= \frac{x + 2(x+1)}{2(x+1)^{\frac{1}{2}}}$$

$$= \frac{3x+2}{2(x+1)^{\frac{1}{2}}}$$

$$\frac{3x+2}{2(x+1)^{\frac{1}{2}}} = -1 \quad \Rightarrow 3x+2=-2(x+1)^{\frac{1}{2}}$$

$$(3x+2)^2 = 4(x+1)$$

$$9x^2 + 12x + 4 = 4x + 4$$

$$9x^2 + 8x = 0$$

$$x(9x+8) = 0$$

$$x = 0 \text{ or } -\frac{8}{9}$$

If $x = 0, \frac{dy}{dx} = 1$ If $x = -\frac{8}{9}, \frac{dy}{dx} = -1$

$\therefore$ $y = x\sqrt{x+1}$ is parallel to the line $x + y = -3$

at $\left(-\frac{8}{9}, -\frac{8}{27}\right)$

10 $\frac{dy}{dx} = \frac{1}{2}(8x^3 - 15x^2 - 10x + 3) = -7$

normal: $y + \frac{5}{2} = \frac{1}{7}(x-1)$

$y = \frac{1}{7}x - \frac{37}{14}$

$\frac{1}{2}(2x^4 - 5x^3 - 5x^2 + 3) = \frac{1}{7}x - \frac{37}{14}$

$14x^4 - 35x^3 - 35x^2 + 21x = 2x - 37$

$14x^4 - 35x^3 - 35x^2 + 19x + 37 = 0$

$x = 3.0782 \quad y = -2.2031 \quad (3.08, -2.20)$

11 $f(x) = [g(x)]^3 \quad \therefore f(0) = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8} \quad \left(0, -\frac{1}{8}\right)$

$f'(x) = 3[g(x)]^2 g'(x)$

$f'(0) = 3\left(-\frac{1}{2}\right)^2\left(\frac{8}{3}\right) = 2$

$y + \frac{1}{8} = 2x$ or $y = 2x - \frac{1}{8}$

12 **a** $y = (1-3x)^7(3x+5)^3$

$\frac{dy}{dx} = (1-3x)^7\, 3(3x+5)^2\, 3 + (3x+5)^3$
$7(1-3x)^6(-3)$
$= 9(1-3x)^7(3x+5)^2 - 21(1-3x)^6$
$(3x+5)^3$
$= 3(1-3x)^6(3x+5)^2[3(1-3x) - 7$
$(3x+5)]$
$= 3(1-3x)^6(3x+5)^2(-30x - 32)$
$= -6(1-3x)^6(3x+5)^2(15x+16)$

b $y = \sqrt{(4x^2 - 3x + 1)^5}$

$\frac{dy}{dx} = \frac{5}{2}(4x^2 - 3x + 1)^{\frac{3}{2}}(8x - 3)$

c $y = \frac{x^2 - 3}{\sqrt{x+1}} \quad x \neq -1$

$$\frac{dy}{dx} = \frac{(x+1)^{\frac{1}{2}}2x - \frac{1}{2}(x+1)^{-\frac{1}{2}}(x^2 - 3)}{(x+1)}$$

$$= \frac{4x(x+1) - (x^2 - 3)}{2(x+1)^{\frac{3}{2}}}$$

$$= \frac{3x^2 + 4x + 3}{2(x+1)^{\frac{3}{2}}}$$

d $y = \sqrt{x + \sqrt{x^2 + 1}} \quad = \left(x + (x^2+1)^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}\left(x + (x^2+1)^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(1 + \frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}}\cancel{2}x\right)$$

$$= \frac{1 + x(x^2+1)^{-\frac{1}{2}}}{2\left(x + (x^2+1)^{\frac{1}{2}}\right)^{\frac{1}{2}}}$$

$$= \frac{(x^2+1)^{\frac{1}{2}} + x}{2\left(x^2+1\right)^{\frac{1}{2}}\left(x + (x^2+1)^{\frac{1}{2}}\right)^{\frac{1}{2}}}$$

$$= \frac{\left[(x^2+1)^{\frac{1}{2}} + x\right]^{\frac{1}{2}}}{2\left(x^2+1\right)^{\frac{1}{2}}}$$

$$= \frac{1}{2}\sqrt{\frac{(x^2+1)^{\frac{1}{2}} + x}{\left(x^2+1\right)}}$$

e $y = (x + 2 + (x-3)^8)^3$

$\frac{dy}{dx} = 3(x + 2 + (x-3)^8)^2(1 + 8(x-3)^7)$

13 $f(x) = ax^3 + 6x^2 - bx$

$f'(x) = 3ax^2 + 12x - b$

$f''(x) = 6ax + 12$

$f''(1) = 0 \quad \therefore 6a + 12 = 0 \quad \therefore a = -2$

$f'(-1) = 0 \quad \therefore -6 - 12 - b = 0 \quad \therefore b = -18$

14 $y = x - 3\sqrt[3]{x} = x - 3x^{\frac{1}{3}}$

a $(0, 0) \quad x^{\frac{1}{3}}(x^{\frac{2}{3}} - 3) = 0$

$x = 0$ or $x^{\frac{2}{3}} = 3$

$x = \pm 27 \; (\sqrt{27}, 0) \; (-\sqrt{27}, 0)$

b $\frac{dy}{dx} = (1 - x^{\frac{-2}{3}})$

$1 - x^{\frac{-2}{3}} = 0 \quad \therefore\ 1 = \frac{1}{x^{\frac{2}{3}}} \quad \therefore\ x^{\frac{2}{3}} = 1 \quad x = \pm 1$

$\frac{d^2y}{dx^2} = \frac{2}{3}x^{\frac{-5}{3}} = \frac{2}{3\sqrt[3]{x^5}}$

If $x = 1$, $\frac{d^2y}{dx^2} > 0 \quad \therefore$ minimum at $(1, -2)$

If $x = -1$, $\frac{d^2y}{dx^2} < 0 \quad \therefore$ maximum at $(-1, 2)$

c $\frac{d^2y}{dx^2} \neq 0 \quad \therefore$ no points of inflexion

d

x	**x < –1**	**x = –1**	**–1 < x < 1**	**x = 1**	**x > 1**
$\frac{dy}{dx}$	+	0	–	0	+

i function increases for $]-\infty, -1[\cup]1, \infty[$

ii function decreases for $]-1, 1[$

15 $y = \frac{2x}{x^2 - 1}$

a $x = 1, x = -1, y = 0$

b $f(-x) = \frac{-2x}{(-x)^2 - 1} = \frac{-2x}{x^2 - 1} = -f(x)$

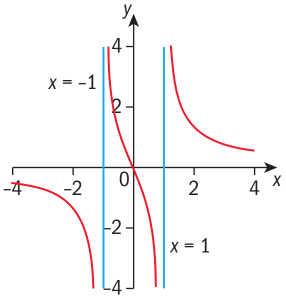
$\therefore$ function is odd

c $\frac{dy}{dx} = \frac{(x^2 - 1)2 - 2x(2x)}{(x^2 - 1)^2}$

$= \frac{-2x^2 - 2}{(x^2 - 1)^2} < 0$

$\therefore\ \frac{dy}{dx} < 0$

d



16 $f(x) = \frac{(x - 3)^2}{x^2 - 3}$

a $(0, -3) \quad (3, 0) \quad x = \pm\sqrt{3} \quad y = 1$

b $f'(x) = \frac{(x^2 - 3)2(x - 3) - 2x(x - 3)^2}{(x^2 - 3)^2}$

$= \frac{2(x - 3)[x^2 - 3 - x(x - 3)]}{(x^2 - 3)^2}$

$= \frac{2(x - 3)(3x - 3)}{(x^2 - 3)^2} = \frac{6(x - 3)(x - 1)}{(x^2 - 3)^2}$

$f'(x) = 0 \quad \Rightarrow \quad x = 3$ or 1

$f'(x) = \frac{6x^2 - 24x + 18}{(x^2 - 3)^2}$

$f''(x) = \frac{(x^2 - 3)^2(12x - 24) - 2(x^2 - 3)2x(6x^2 - 24x + 18)}{(x^2 - 3)^4}$

$= \frac{(x^2 - 3)(12x - 24) - 4x(6x^2 - 24x + 18)}{(x^2 - 3)^3}$

$= \frac{-12x^3 + 72x^2 - 108x + 72}{(x^2 - 3)^3}$

If $x = 3$, $f''(x) > 0 \quad \therefore$ minimum at $(3, 0)$

If $x = 1$, $f''(x) < 0 \quad \therefore$ maximum at $(1, -2)$

c $f''(x) = 0 \quad \Rightarrow \quad x = 4.1958$

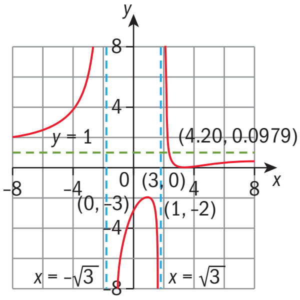
x	**x < 4.1958**	**x > 4.1958**
f″(x)	+	–

$\therefore$ point of inflexion at $(4.20, 0.0979)$

d **i** increasing for $]-\infty, -\sqrt{3}[\cup]-\sqrt{3}, 1[\cup]3, \infty[$

ii decreasing for $]1, \sqrt{3}[\cup]\sqrt{3}, 3[$

e



17 $x = y^5 - y \quad 0 = y(y^4 - 1)$

$y = 0, \pm 1\ (0, 0)\ (0, 1)\ (0, -1)$

$\frac{dx}{dy} = 5y^4 - 1 \quad \therefore\ \frac{dy}{dx} = \frac{1}{5y^4 - 1}$

At $(0, 0)$ $\frac{dy}{dx} = -1$

At $(0, 1)$ $\frac{dy}{dx} = \frac{1}{4}$

Review exercise

1 $(1.5, 0) \quad (x, \sqrt{x}) \quad$ Let l = distance

$l^2 = (x - 1.5)^2 + x$

$= x^2 - 2x + 2.25$

$l = (x^2 - 2x + 2.25)^{\frac{1}{2}}$

$\frac{dl}{dx} = \frac{1}{2}(x^2 - 2x + 2.25)^{\frac{1}{2}}(2x - 2)$

$= \frac{x - 1}{(x^2 - 2x + 2.25)^{\frac{1}{2}}}$

$\frac{dl}{dx} = 0 \Rightarrow x = 1$

x	$x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	−	0	+

$\therefore$ minimum distance when $x = 1$

minimum distance $= \sqrt{1.25}$

2 Let r = radius, x = side of square

$4\pi r + 4x = 80$

$x = 20 - \pi r$

$A = 2\pi r^2 + x^2$

$= 2\pi r^2 + (20 - \pi r)^2$

$= 2\pi r^2 + 400 - 40\pi r + \pi^2 r^2$

$\frac{dA}{dr} = 4\pi r - 40\pi + 2\pi^2 r$

$\frac{dA}{dr} = 0 \Rightarrow 4\pi r + 2\pi^2 r = 40\pi$

$2r + \pi r = 20$

$r = \frac{20}{2 + \pi}$

$\frac{d^2 A}{dr^2} = 4\pi + 2\pi^2 > 0 \therefore$ minimum

$r = \frac{20}{2 + \pi}$

3 $\frac{dr}{dt} = 3$ cm min^{-1} $\quad \frac{dh}{dt} = -4$ cm min^{-1}

$v = \pi r^2 h$

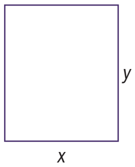
$\frac{dv}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi rh \frac{dr}{dt}$

$= \pi(81)(-4) + 2\pi(9)(12)(3)$

$= 324\pi$ cm^3 min^{-1}

increasing at a rate of 324π cm^3 m^{-1}

4



$xy = 180 \quad y = \frac{180}{x}$

printing area, $A = (x - 2)(y - 3)$

$A = xy - 3x - 2y + 6$

$= 180 - 3x - \frac{360}{x} + 6$

$\frac{dA}{dx} = -3 + \frac{360}{x^2}$

$\frac{dA}{dx} = 0 \Rightarrow 3x^2 = 360$

$x^2 = 120$

$x = \sqrt{120} = 10.95 \quad y = 16.43$

$\frac{d^2 A}{dx^2} = \frac{-720}{x^3} < 0 \therefore$ maximum

$\therefore$ dimensions are 11.0 cm by 16.4 cm

5 $\frac{dx}{dt} = \frac{1}{1 + 2x} = (1 + 2x)^{-1}$

acceleration $= \frac{d^2 x}{dt^2} = -(1 + 2x)^{-2}\, 2\frac{dx}{dt}$

$= \frac{-2}{(1 + 2x)^3}$

At $x = 2$, acceleration $= \frac{-2}{125}$

5

Aesthetics in mathematics

Answers

Skills check

1 $f(x) = \frac{x}{x-1},\ x \pm 1$

inverse: $x = \frac{y}{y-1}$

$x(y-1) = y$

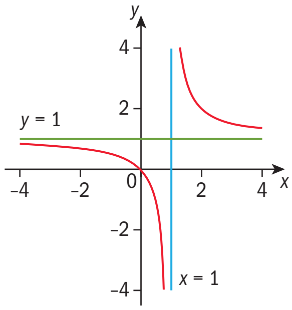
$xy - y = x$

$y(x-1) = x$

$y = \frac{x}{x-1}$

$f^{-1}(x) = \frac{x}{x-1},\ x \neq 1$

$\therefore f^{-1}(x) = f(x)$



2 $f(x) = ax - b \quad g(x) = \frac{x+b}{a}$

$f \circ g(x) = f\left(\frac{x+b}{a}\right) = a\left(\frac{x+b}{a}\right) - b = x$

$g \circ f(x) = g(ax - b) = \frac{ax - b + b}{a} = x$

$\therefore f \circ g(x) = g \circ f(x)$

Exercise 5A

1 $u_1 = 1 \qquad u_n = \frac{u_{n+1}}{1 + u_{n-1}} \qquad n \in \mathbb{Z}^+$

$u_2 = \frac{1}{2} \qquad u_3 = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$

$u_4 = \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{1}{4} \qquad u_5 = \frac{\frac{1}{4}}{\frac{5}{4}} = \frac{1}{5}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \qquad u_n = \frac{1}{n}$

2 $u_1 = 2 \quad u_n = \frac{u_{n-1}}{1 + u_{n-1}}$

$n \in \mathbb{Z}^+,\ n \geq 2$

$u_2 = \frac{2}{-1} = -2 \quad u_3 = \frac{-2}{3}$

$u_4 = \frac{\frac{-2}{3}}{\frac{5}{3}} = \frac{-2}{5} \qquad u_5 = \frac{\frac{-2}{5}}{\frac{7}{5}} = \frac{-2}{7}$

$u_6 = \frac{\frac{-2}{7}}{\frac{9}{7}} = \frac{-2}{9} \quad 2, -2, \frac{-2}{3}, \frac{-2}{5}, \frac{-2}{7}, \frac{-2}{9} \quad u_n = \frac{-2}{2n-3}$

3 **a** $u_0 = 2 \quad u_n = u_{n-1} = \frac{1}{2}n \quad n \in \mathbb{Z}^+$

$u_1 = 2 - \frac{1}{2} = \frac{3}{2} \qquad u_2 = \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$

$u_3 = \frac{5}{4} - \frac{1}{8} = \frac{9}{8} \quad u_4 = \frac{9}{8} - \frac{1}{16} = \frac{17}{6}$

$u_5 = \frac{17}{16} - \frac{1}{32} = \frac{33}{32} \quad 2, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \frac{17}{16}, \frac{33}{32}$

b $u_n = \frac{2^n + 1}{2^n}$

c $P(n): u_n = \frac{2^n + 1}{2^n}$

$P(0) \Rightarrow \quad u_0 = \frac{2^0 + 1}{2^0} = \frac{2}{1} = 2$

$\therefore P(0)$ is true

Assume $P(k)$ $u_k = \frac{2^k + 1}{2^k}$

Prove $P(k+1)$ $u_{k+1} = u_k - \frac{1}{2^{k+1}}$

$$= \frac{2^k + 1}{2^k} - \frac{1}{2^{k+1}}$$

$$= \frac{2(2^k + 1) - 1}{2^{k+1}}$$

$$= \frac{2^{k+1} + 2 - 1}{2^{k+1}}$$

$$= \frac{2^{k+1} + 1}{2^{k+1}}$$

$\therefore P(k) \Rightarrow P(k+1)$ and $P(0)$ is true

$\therefore$ by mathematical induction, $u_n = \frac{2^n + 1}{2^n}$

4 **a** $u_1 = 1 \quad u_n = u_{n-1} + 2_n - 3, \quad n \in \mathbb{Z}$

$u_2 = 1 + 4 - 3 = 2$

$u_3 = 2 + 6 - 3 = 5$

$u_4 = 5 + 8 - 3 = 10$

$u_5 = 10 + 10 - 3 = 17 \quad 1, 2, 5, 10, 17$

b $P(n) : un = n^2 - 2n + 2$

$P(1) \Rightarrow u_1 = 1^2 - 2(1) + 2 = 1 \quad \therefore P(1)$ is true

Assume $P(k) \quad u_k = k^2 - 2k + 2$

Prove $P(k+1) \;\; u_{k+1} = u_k + 2(k+1) - 3$

$= k^2 - 2k + 2 + 2k + 2 - 3$

$= k^2 + 1$

$(k+1)^2 - 2(k+1) + 2 = k^2 + 2k + 1$

$- 2k - 2 + 2$

$= k^2 + 1$

$\therefore u_{k+1} = (k+1)^2 - 2(k+1) + 2$

$\therefore P(k) \Rightarrow P(k+1)$ and $P(1)$ is true

$\therefore$ by mathematical induction, $u_n = n^2 - 2n + 2$

Exercise 5B

1 **a** $(64)^{\frac{2}{3}} = 4^2 = 16$

b $\left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{2}{3}$

c $\left(\frac{81}{16}\right)^{\frac{-3}{4}} = \left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

2 **a** $\left(\frac{b^{-3}x^{-2}}{8x}\right)^{\frac{-2}{3}} = \left(\frac{8x}{b^{-3}x^{-2}}\right)^{\frac{2}{3}} = (8x^3b^3)^{\frac{2}{3}}$

$\therefore \left(\frac{b^{-3}x^{-2}}{8x}\right)^{\frac{-2}{3}} = (2xb)^2 = 4b^2x^2$

b $\frac{a^{-1} - a^{-2}}{a^{-3}} = \frac{a^{-1}}{a^{-3}} - \frac{a^{-2}}{a^{-3}} = a^2 - a = a(a-1)$

c $\frac{x^3 \times x^{-7}}{x^{-4}} = \frac{x^{-4}}{x^{-4}} = 1$

3 $\sqrt{y^3} \div \sqrt[3]{y^2} = y^{\frac{3}{2}} \div y^{\frac{2}{3}} = y^{\left(\frac{3}{2}-\frac{2}{3}\right)} = y^{\frac{5}{6}}$

when $y = 64$, $y^{\frac{5}{6}} = \left(\sqrt[6]{64}\right)^5 = 2^5 = 32$

4 $\frac{(x^4yz^{-3})^2 \times \sqrt{x^{-5}y^2z}}{(xz)^{\frac{7}{2}}} = \frac{x^8y^2z^{-6} \times x^{-\frac{5}{2}}yz^{\frac{1}{2}}}{x^{\frac{7}{2}}z^{\frac{7}{2}}}$

$= \frac{x^{\frac{11}{2}}y^3z^{-\frac{11}{2}}}{x^{\frac{7}{2}}z^{\frac{7}{2}}} = x^2\,y^3\,z^{-9}$

$= \frac{x^2y^3}{z^9}$

5 $5 \times 4^{3n+1} - 20 \times 8^{2n} = 5 \times (2^2)^{3n+1} - 20 \times (2^3)^{2n}$

$= 5 \times 2^{6n+2} - 20 \times 2^{6n}$

$= 5 \times 2^2 \times 2^{6n} - 20 \times 2^{6n}$

$= 20 \times 2^{6n} - 20 \times 2^{6n}$

$= 0$

6 $4^x + 2 = 3 \times 2^x$

$(2^x)^2 - 3(2^x) + 2 = 0$

$(2^x - 1)(2^x - 2) = 0$

$2^x = 1$ or $2^x = 2$

$x = 0$ or 1

Exercise 5C

1 $250000\,(1 + r)^{20} = 450000$

$(1 + r)^{20} = 1.8$

$1 + r = 1.0298$

$r = 0.0298 = 2.98\% = 3\%$ (nearest percent)

2 **a** $61.08 = 17.48\,(1 + r)^7$

$(1 + r)^7 = 3.494$

$1 + r = 1.1957$

$r = 0.196 = 19.6\%$

b $77.45 = 61.08\,(1 + r)^4$

$(1 + r)^4 = 1.268\ldots$

$1 + r = 1.0612$

$r = 0.0612 = 6.12\%$

c $97.87 = 72.99\,(1 + r)^{12}$

$(1 + r)^{12} = 1.3408$

$1 + r = 1.0247$

$r = 0.0247 = 2.47\%$

3 Samira: After 15 years (60 quarters), she will have

$1000\left(1+\frac{0.08}{4}\right)^{60} = 1000 \times (1.02)^{60}$

$\simeq \$\ 3280$

Hemanth: $500(1.08)^{15} + 500\left(1+\frac{0.084}{12}\right)^{12\times15}$

$= 500(1.08)^{15} + 500\,(1.007)^{180}$

$= 1586.08 + 1754.99$

$\simeq \$\ 3340$

4

End of year	Amount owing	Pays back
1	15000 × 1.05 = 15750	10500
2	5250 × 1.05 = 5512.5	3675
3	1837.5 × 1.05 = 1929.375	1286.25
4	643.125 × 1.05 = 675.28	450.1875
5	225.09375 × 1.05 = 236.348	157.565625
		16069.00313

$\therefore$ Guiseppe has paid back approx. € 16700

Exercise 5D

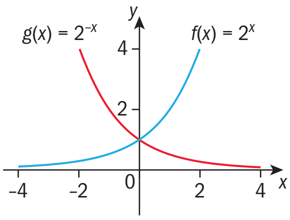
1 Red curve: (1,2.5) $\quad 2.5 = a^1 \quad \therefore a = 2.5$

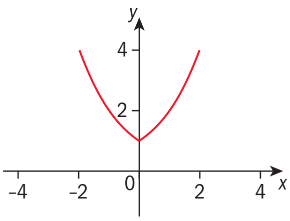
Blue curve: (−1, 4) $\quad 4 = a^{-1} \quad \therefore a = \frac{1}{4}$

2 $f(x) = 2^x \quad f(x+1) = 2^{(x+1)} = 2(2^x) = 2f(x)$

$f(x+a) = 2^{x+a} = 2^a(2^x) = 2^a f(x)$

3





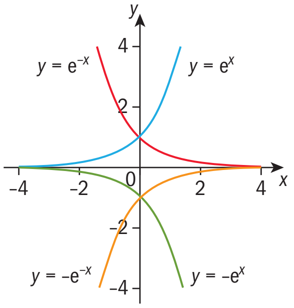
4 $e^x + 1 = e^{x+1}$

$x = -0.541$

5 **a** reflection in the y-axis

b reflection in the x-axis

c rotation of 180° about the origin (or reflection is the y-axis followed by reflection in the x-axis)



Exercise 5E

1 **a** $5^3 = 125 \Rightarrow \log_5 125 = 3$

b $10^3 = 1000 \Rightarrow \log_{10} 1000 = 3$

c $27^{\frac{1}{3}} = 3 \Rightarrow \log_{27} 3 = \frac{1}{3}$

d $10^{-3} = 0.001 \Rightarrow \log_{10} 0.001 = -3$

e $m = n^2 \Rightarrow \log_n m = 2$

f $a^b = 2 \Rightarrow \log_a 2 = b$

2 **a** $\log_3 9 = 2 \Rightarrow 3^2 = 9$

b $\log_{10} 1000000 = 6 \Rightarrow 10^6 = 1000000$

c $\log_{49} 7 = \frac{1}{2} \Rightarrow 49^{\frac{1}{2}} = 7$

d $\log_a 1 = 0 \Rightarrow a^0 = 1$

e $\log_4 a = b \Rightarrow 4^b = a$

f $\log_p q = r \Rightarrow p^r = q$

3 **a** $\log_8 64 = 2$

b $\log_9 3 = \frac{1}{2}$

c $\log_{10} 0.01 = -2$

d $\log_{144} 12 = \frac{1}{2}$

e $\log_{37} 1 = 0$

f $\log_a \sqrt[3]{a} = \frac{1}{3}$

4 **a** $\log_x 81 = 2 \quad \therefore x^2 = 81 \quad \therefore x = 9$

b $\log_3 x = 4 \quad \therefore 3^4 = x \quad \therefore x = 81$

c $\log_{11} 121 = x \quad \therefore x = 2$

d $\log_x 5 = \frac{1}{3} \quad \therefore x^{\frac{1}{3}} = 5 \quad \therefore x = 125$

e $\log_x 16 = \frac{2}{3} \quad \therefore x^{\frac{2}{3}} = 16 \quad \therefore x = 16^{\frac{3}{2}} = 4^3 = 64$

f $\log_x 32 = -5 \quad \therefore x^{-5} = 32 \quad \therefore \frac{1}{x^5} = 32,\ x^5 = \frac{1}{32},$

$x = \frac{1}{2}$

Exercise 5F

1 **a** $\log_a \frac{p^2}{q} = 2\log_a p - \log_a q$

b $\log_a \sqrt[3]{\frac{p}{q^2}} = \frac{1}{3}\log_a \frac{p}{q^2} = \frac{1}{3}\ \log_a p - \frac{2}{3}\log_a q$

2 **a** $\log 4 + 2\log 3 - \log 6 = \log \frac{4 \times 9}{6} = \log 6$

b $\frac{1}{2}\log_a p + \frac{1}{4}\log_a q^2 = \log_a p^{\frac{1}{2}} + \log_a q^{\frac{1}{2}} = \log_a \sqrt{pq}$

c $2 - \log 5 = \log 100 - \log 5 = \log 20$

3 **a** $\log 5 + \log 8 - \log 4 = \log \frac{5 \times 8}{4} = \log 10 = 1$

b $\log_2 48 - \frac{1}{3}\log_2 27 = \log_2 48 - \log_2 3 = \log_2 16 = 4$

c $2 + \log_5 10 - \log_5 2 = 2 + \log_5 5 = 2 + 1 = 3$

4 **a** $3\log y = 2\log x \Rightarrow y^3 = x^2 \ \therefore\ y = x^{\frac{2}{3}}$

b $\log y = \log x + \log 2 \Rightarrow y = 2x$

c $\log y - 3\log x = \log 2 \Rightarrow \frac{y}{x^3} = 2 \ \therefore\ y = 2x^3$

d $\log y = 2 + 3x \Rightarrow y = 10^{2+3x}$

Exercise 5G

1 **a** $\log_3 2 \times \log_3 81 = \log_3 2 \times \frac{\log_3 81}{\log_3 2} = 4$

b $\log_6 10 \times \log 6 = \log_6 10 \times \frac{\log_6 6}{\log_6 10} = 1$

c $\log_{125} 8 \times \log_8 5 = \log_{125} 8 \times \frac{\log_{125} 5}{\log_{125} 8} = \frac{1}{3}$

d $\frac{1}{\log_2 6} + \frac{1}{\log_3 6} = \frac{1}{\log_2 6} + \frac{\log_2 3}{\log_2 6}$

$= \frac{\log_2 2 + \log_2 3}{\log_2 6} = \frac{\log_2 6}{\log_2 6} = 1$

e $\frac{1}{\log_4 6} + \frac{1}{\log_9 6} = \frac{1}{\log_4 6} + \frac{\log_4 9}{\log_4 6} = \frac{\log_4 4 + \log_4 9}{\log_4 6}$

$= \frac{\log_4 36}{\log_4 6} = \frac{2\log_4 6}{\log_4 6} = 2$

f $\log_5 40 - \frac{1}{\log_8 5} = \log_5 40 - \frac{\log_5 8}{\log_5 5} = \log_5 5 = 1$

2 **a** let $a^{\log b} = x \Rightarrow \log_a x = \log b$

$\frac{\log x}{\log a} = \log b$

$\frac{\log x}{\log b} = \log a$

$\log_b x = \log a$

$\therefore x = b^{\log a}$

$\therefore a^{\log b} = b^{\log a}$

b $\frac{1}{\log_a ab} + \frac{1}{\log_b ab} = \frac{\log_{ab} a}{\log_{ab} ab} + \frac{\log_{ab} b}{\log_{ab} ab}$

$= \log ab^a + \log ab^b$

$= \log ab^{(ab)}$

$= 1$

$\therefore \frac{1}{\log_a ab} + \frac{1}{\log_b ab} = 1$

3 $p = \log a^x \quad q = \log a^y$

$\log_x a = \frac{1}{\log_a x} = \frac{1}{p} \quad = \log_y a = \frac{1}{\log_a y} = \frac{1}{q}$

a $\log_{xy} a = \frac{\log_a a}{\log_a xy} = \frac{1}{\log_a x + \log_a y} = \frac{1}{p+q}$

b $\log_{xy} a = \frac{1}{\log_a \left(\frac{x}{y}\right)} = \frac{1}{\log_a x - \log_a y} = \frac{1}{p-q}$

Exercise 5H

1 **a** $5^x = 7$

$x\log 5 = \log 7$

$x = \frac{\log 7}{\log 5}$

$x = 1.21$

b $4^{2x-1} = 3$

$(2x - 1)\log 4 = \log 3$

$2x - 1 = \frac{\log 3}{\log 4}$

$x = \frac{1}{2}\left(\frac{\log 3}{\log 4} + 1\right)$

$\therefore x = 0.896$

2 $(2^x)(5^x) = 0.01$

$10^x = 0.01$

$\therefore x = -2$

Exercise 5I

1 **a** $2^{3x} = 5$

$3x\log 2 = \log 5$

$x = \frac{\log 5}{3\log 2}$

$x = 0.774$

b $3^x(3^{x-1}) = 10$

$3^{2x-1} = 10$

$(2x - 1)\log 3 = \log 10$

$2x - 1 \quad \frac{1}{\log 3}$

$x = \frac{1}{2}\left(\frac{1}{\log 3} + 1\right)$

$x = 1.55$

2 **a** $4\log_3 x = \log_x 3$

$4\log_3 x = \frac{1}{\log_3 x}$

$(\log_3 x)^2 = \frac{1}{4} \quad \therefore \log_3 x = \pm\frac{1}{2}$

$x = 3^{\frac{1}{2}}$ or $x = 3^{-\frac{1}{2}}$

b $3\log_2 x + \log_2 27 = 3$

$\log_2(27x^3) = 3$

$\therefore 27x^3 = 8$

$x^3 = \frac{8}{27}$

$x = \frac{2}{3}$

3 $9^x - 6(3^x) - 16 = 0$

$(3^x)^2 - 6(3^x) - 16 = 0$

$(3^x - 8)(3^x + 2) = 0$

$3^x = 8$

$x\log 3 = \log 8$, $x = 1.89$

4 $\log_4 x + 12\log_x 4 - 7 = 0$

$\log_4 x + \frac{12}{\log_4 x} - 7 = 0$

$(\log_4 x)^7 - 7\log_4 x + 12 = 0$

$(\log_4 x - 3)(\log_4 x - 4) = 0$

$\log_4 x = 3$ or $\log_4 x = 4$

$x = 4^3$ or $x = 4^4$

$x = 64$ or 256

5 $5^{x+1} + \frac{4}{5^x} - 21 = 0$

$5(5^x)^2 - 21(5^x) + 4 = 0$

$(5(5^x) - 1)(5^x - 4) = 0$

$5^x = \frac{1}{5}$ or $5^x = 4$

$x = -1$ or $x\log 5 = \log 4$

$x = -1$ or 0.861

6 $\log_3 x + \log_x 9 - 3 = 0$

$\log_3 x + \frac{\log_3 9}{\log_3 x} - 3 = 0$

$(\log_3 x)^2 - 3\log_3 x - 2 = 0$

$(\log_3 x - 2)(\log_3 x - 1) = 0$

$\log_3 x = 2$ or $\log_3 x = 1$

$x = 3^2$ or $x = 3^1$

$x = 3$ or 9

7 $3 \times 9^x - 2 \times 4^x = 5 \times 6^x$

$3(3^x)^2 - 5(2^x)(3^x) - 2(2^x)^2 = 0$

$(3(3^x) + 2^x)(3^x - 2(2^x)) = 0$

$3(3^x) = -2^x$ or $3^x = 2(2^x)$

$3(1.5)^x = -1$ or $1.5^x = 2$

No solution, $x\log 1.5 = \log 2$

$x = 1.71$

8 $6\log_2 x + 6\log_8 y = 7$

$6\log_2 x + \frac{6\log_2 y}{\log_2 8} = 7$

$6\log_2 x + 2\log_2 y = 7$

$\log_2 x^6y^2 = 7$

$x^6y^2 = 2^7 \quad \therefore x^6y^2 = 128 \qquad (1)$

$4\log_4 x + 4\log_2 y = 9$

$\frac{4\log_2 x}{\log_2 4} + 4\log_2 y = 9$

$2\log_2 x + 4\log_2 y = 9$

$\log_2 x^2y^4 = 9$

$\therefore x^2y^4 = 2^9 \quad \therefore x^2y^2 = 512 \ (2)$

from (1) $y^2 = \frac{128}{x^6} \qquad x^2\left(\frac{128}{x^6}\right)^2 = 512$

$16384 = 512x^{10} \qquad \therefore x^{10} = 32$

$x = \sqrt{2}$

$y^2 = \frac{128}{8} = 16 \qquad y = 4$

9 $2\log xy = 1 \Rightarrow x = y^2$

$xy = 125 \quad \therefore y^3 = 125$

$y = 5, x = 25$

10 $y\log_2 8 = x \Rightarrow 3y = x$

$2^x + 8^y = 64$

$2^{3y} + 2^{3y} = 64$

$2^{3y+1} = 26$

$3y + 1 = 6$

$y = \frac{5}{3}, \quad x = 5$

11 a $\log_5 x = y = \log_{25}(2x - 1)$

$x = 5^y \quad 25^y = 2x - 1$

$(5^y)^2 = 2x - 1$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1, y = 0$

b $\log(x + y) = 0 \Rightarrow x + y = 1 \Rightarrow y = 1 - x$

$2\log x = \log(y + 5) \Rightarrow x^2 = y + 5$

$x^2 = 1 - x + 5$

$x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

$x = 2 \quad y = -1$

(x cannot be negative)

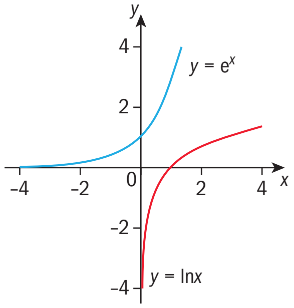
Exercise 5J

1 $f(x) = \mathrm{e}^x$: domain is $x \in \mathbb{R}$

range is $y \in \mathbb{R}, y > 0$

$f^{-1}(x) = \ln x$: domain is $x \in \mathbb{R}, x > 0$

range is $y \in \mathbb{R}$



2 $f(x) = a^x \qquad y = a^x$

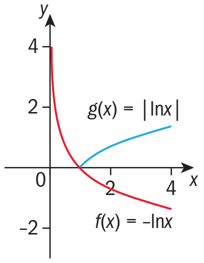
Inverse : $x = a^y, \quad y = \log_a x$

$\therefore f^{-1}(x) = \log_a x$

$f \circ f^{-1}(x) = f(\log a^x) = a^{\log ax}$

$\therefore a^{\log ax} = x$

3



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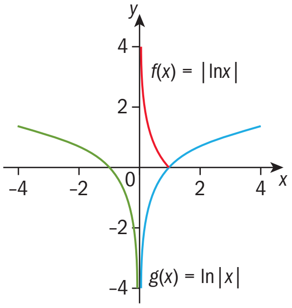

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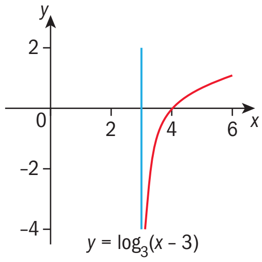
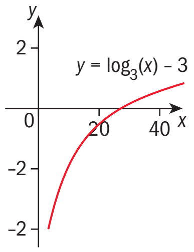


translation of $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$



translation of $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

6 a $y = \ln(x - 1) - 1$

$x - 1 > 0 \quad \therefore x > 1$

domain : $x \in \mathbb{R}, x > 1$

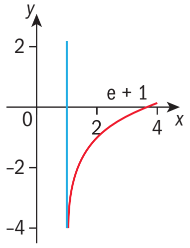
asymptote : $x = 1$

x-intercept $0 = \ln(x - 1) - 1$

$\ln(x - 1) = 1$

$(\mathrm{e} + 1, 0) \quad x - 1 = \mathrm{e}$

$x = \mathrm{e} + 1$



b $y = \log_3(9 - 3x) + 2$

$9 - 3x > 0$

$9 > 3x$

$x < 3$

domain : $x \in \mathbb{R}, x < 3$

asymptote : $x = 3$

x-intercept $0 = \log_3(9 - 3x) + 2$

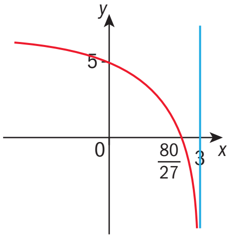
$-2 = \log_3(9 - 3x)$

$3^{-2} = 9 - 3x$

$3x = 9 - \frac{1}{9} = \frac{80}{9}$

$x = \frac{80}{27} \quad \left(\frac{80}{27}, 0\right)$

y-intercept $y = \log_3 9 + 2 = 4(0, 4)$



Exercise 5K

1 **a** $y = \frac{3}{2}\,\mathrm{e}^{x^2} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{3}{2}(2x\mathrm{e}^{x^2}) = 3x\mathrm{e}^{x^2}$

b $y = \frac{-5}{\mathrm{e}^{3x-1}} = -5\mathrm{e}^{-(3x-1)} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 15\mathrm{e}^{-(3x-1)} = \frac{15}{\mathrm{e}^{3x-1}}$

c $y = \mathrm{e}^{4x-1} + 4 \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 4\mathrm{e}^{4x-1}$

d $y = \mathrm{e}^x + \frac{1}{\mathrm{e}^x} = \mathrm{e}^x + \mathrm{e}^{-x} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x - \mathrm{e}^{-x} = \mathrm{e}^x - \frac{1}{\mathrm{e}^x}$

e $y = \mathrm{e}^{-(1-3x)} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^{-(1-3x)}$

f $y = 2\mathrm{e}^{\sqrt{x}} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\mathrm{e}^{\sqrt{x}} = \frac{\mathrm{e}^{\sqrt{x}}}{\sqrt{x}}$

2 **a** $y = x\mathrm{e}^x \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x + x\mathrm{e}^x$

b $y = \frac{x^2}{\mathrm{e}^x} = x^2\mathrm{e}^{-x} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 2x\mathrm{e}^{-x} - x^2\mathrm{e}^{-x} = \frac{2x - x^2}{\mathrm{e}^x}$

c $y = \frac{\mathrm{e}^{2x}}{\sqrt{x}} = \mathrm{e}^{2x}x^{-\frac{1}{2}} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2x}x^{-\frac{1}{2}} - \mathrm{e}^{2x}\frac{1}{2}x^{-\frac{3}{2}}$

$= \frac{2\mathrm{e}^{2x}}{x^{-\frac{1}{2}}} - \frac{\mathrm{e}^{2x}}{2x^{\frac{3}{2}}} = \frac{4x\mathrm{e}^{2x} - \mathrm{e}^{2x}}{2x^{\frac{3}{2}}}$

d $y = \sqrt{x}\,\mathrm{e}^{\sqrt{x}} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^{-\frac{1}{2}}\mathrm{e}^{\sqrt{x}} + \sqrt{x}\,\frac{1}{2}x^{-\frac{1}{2}}\mathrm{e}^{\sqrt{x}}$

$= \frac{\mathrm{e}^{\sqrt{x}}}{2\sqrt{x}} + \frac{1}{2}\mathrm{e}^{\sqrt{x}}$

3 **a** $y = \frac{\mathrm{e}^{2x}}{\sqrt{x}} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{x}\mathrm{e}^{2x} - \mathrm{e}^{2x}\frac{1}{2}x^{-\frac{1}{2}} = \frac{4x\mathrm{e}^{2x} - \mathrm{e}^{2x}}{2x^{\frac{3}{2}}}$

b $y = \frac{1 - x^2}{\mathrm{e}^x} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x(-2x) - (1 - x^2)\mathrm{e}^x}{\mathrm{e}^{2x}} = \frac{x^2 - 2x - 1}{\mathrm{e}^x}$

c $y = \frac{\mathrm{e}^{3x}}{1 + x} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1 + x)3\mathrm{e}^{3x} - \mathrm{e}^{3x}}{(1 + x)^2} = \frac{\mathrm{e}^{3x}(2 + 3x)}{(1 + x)^2}$

d $y = \frac{1 + \mathrm{e}^x}{1 - \mathrm{e}^x} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1 - \mathrm{e}^x)\mathrm{e}^x - (1 + \mathrm{e}^x)(-\mathrm{e}^x)}{(1 - \mathrm{e}^x)^2} = \frac{2\mathrm{e}^x}{(1 - \mathrm{e}^x)^2}$

4 **a** $y = \frac{x\mathrm{e}^x}{1 + \mathrm{e}^x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1 + \mathrm{e}^x)(x\mathrm{e}^x + \mathrm{e}^x) - x\mathrm{e}^x(\mathrm{e}^x)}{(1 + \mathrm{e}^x)^2} = \frac{x\mathrm{e}^x + \mathrm{e}^x + \mathrm{e}^{2x}}{(1 + \mathrm{e}^x)^2}$

$= \frac{\mathrm{e}^x(x + 1 + \mathrm{e}^x)}{(1 + \mathrm{e}^x)^2}$

b $y = (1 + \mathrm{e}^x)^2 \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^x(1 + \mathrm{e}^x)$

c $y = \sqrt{1 + \mathrm{e}^{-x}} = (1 + \mathrm{e}^{-x})^{\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(1 + \mathrm{e}^{-x})^{-\frac{1}{2}}(-\mathrm{e}^{-x}) = \frac{-\mathrm{e}^{-x}}{2\sqrt{1 + \mathrm{e}^{-x}}}$

d $y = \frac{x + \mathrm{e}^x}{\mathrm{e}^{-x}} = \mathrm{e}^x(x + \mathrm{e}^x)$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x(1 + \mathrm{e}^x) + \mathrm{e}^x(x + \mathrm{e}^x)$

$= \mathrm{e}^x(1 + x + 2\mathrm{e}^x)$

e $y = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\mathrm{e}^x + \mathrm{e}^{-x})(\mathrm{e}^x - \mathrm{e}^{-x}) - (\mathrm{e}^x + \mathrm{e}^{-x})(\mathrm{e}^x + \mathrm{e}^{-x})}{(\mathrm{e}^x - \mathrm{e}^{-x})^2}$

$= \frac{-4}{(\mathrm{e}^x - \mathrm{e}^{-x})^2}$

5 $f(x) = x\,\mathrm{e}^x \quad -3 \le x \le 3$

a $f'(x) = x\,\mathrm{e}^x + \mathrm{e}^x$

$\mathrm{e}^x(x + 1) = 0 \quad \therefore x = -1 \quad y = -\mathrm{e}^{-1}$

$\therefore$ one stationary point at $\left(-1, \frac{-1}{\mathrm{e}}\right)$

$f''(x) = \mathrm{e}^x + \mathrm{e}^x(x + 1)$

$f''(-1) = \mathrm{e}^{-1} > 0$ minimum

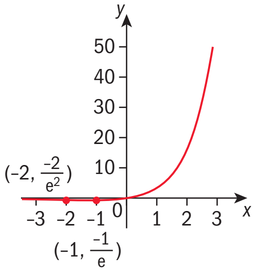
b For point of inflexion $f''(x) = 0$

$\therefore \mathrm{e}^x(x + 2) = 0 \quad \therefore x = -2 \quad y = -2\mathrm{e}^{-2}$

x	-3	-2	-1
f''(x)	$-\mathrm{e}^{-3} < 0$	0	$\mathrm{e}^{-1} > 0$

$\therefore$ change of sign

$\therefore$ point of inflexion at $\left(-2, \frac{-2}{\mathrm{e}^2}\right)$



c At point of inflexion $\left(-2, \frac{-2}{\mathrm{e}^2}\right)$

$f'(-2) = -\mathrm{e}^{-2} = -\frac{1}{\mathrm{e}^2}$

Equation of tangent: $y + \left(-2, \frac{-2}{\mathrm{e}^2}\right) = -\frac{1}{\mathrm{e}^2}(x + 2)$

$y = -\frac{1}{\mathrm{e}^2}(x + 4)$

d $y = 0, x = -4 \quad (-4, 0)$

e y-intercept $= \left(0, \frac{-4}{\mathrm{e}^2}\right)$ area $= \frac{1}{2} \times 4 \times \frac{4}{\mathrm{e}^2} = \frac{8}{\mathrm{e}^2}$

Exercise 5L

1 **a** $y = 5^{3x} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = (3\ln 5)\,5^{3x}$

b $y = \ln(4x + 1) \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{4x + 1}$

2 **a** $y = 1 + 2\ln x \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$

b $y = \frac{1}{\ln x} = (\ln x)^{-1} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -(\ln x)^{-2}\frac{1}{x} = \frac{-1}{x(\ln x)^2}$

Exercise 5M

1 **a** $y = x^2 \ln x \quad \frac{dy}{dx} = x^2\left(\frac{1}{x}\right) + 2x \ln x = x + 2x \ln x$

b $y = xa^x \quad \frac{dy}{dx} = (x \ln a)a^x + a^x = a^x(x \ln a + 1)$

2 **a** $y = \ln\left(\frac{1}{x}\right) = -\ln x \quad \frac{dy}{dx} = \frac{-1}{x}$

b $y = \ln x^2 = 2 \ln x \quad \frac{dy}{dx} = \frac{2}{x}$

c $y = \frac{\ln x}{x} \quad \frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

3 **a** $x^y = e^x$

$y \ln x = x$

$y = \frac{x}{\ln x} \quad \frac{dy}{dx} = \frac{\ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$

b $y = x^{2x}$

$\ln y = 2x \ln x$

$\frac{1}{y}\frac{dy}{dx} = 2x\left(\frac{1}{x}\right) + 2 \ln x$

$\frac{dy}{dx} = x^{2x}(2 + 2 \ln x)$

4 **a** $y = e^x(x - 1)$

$\frac{dy}{dx} = e^x + e^x(x - 1) = xe^x$

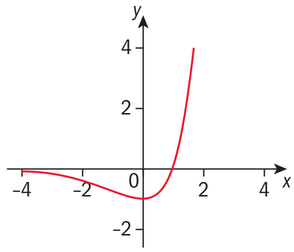
$\frac{dy}{dx} = 0$ if $x = 0$ $\therefore$ only one stationary point

b $\frac{d^2y}{dx^2} = xe^x + e^x$

if $x = 0$, $\frac{d^2y}{dx^2} = 1 > 0$ $\quad\therefore$ minimum at $(0,-1)$

c $(1, 0)$

d



5 **a** $x \in \mathbb{R}$

b $f(-x) = \ln(1 + (-x)^2)$

$= \ln(1 + x^2) = f(x)$

$\therefore$ the y-axis is a line of symmetry

c $f(x) = \ln(1 + x^2)$

$f'(x) = \frac{2x}{1+x^2} = 0$ if $x = 0$

$f''(x) = \frac{(1+x^2)2 - 2x(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$

$f''(0) = 2 > 0$ $\quad\therefore$ minimum at $(0, 0)$

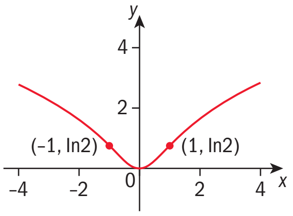
$f''(x) = 0 \Rightarrow \quad x = \pm 1$

x	-2	-1	-0	1	2
$f'(x)$	$\frac{-6}{25} < 0$	0	$2 > 0$	0	$0\ \frac{-6}{25} < 0$

$\therefore$ change of sign

$\therefore$ point of inflexion at $(-1, \ln 2)$ and $(1, \ln 2)$

d



e At $(1, \ln 2)$, $f'(1) = 1$

tangent: $y - \ln 2 = x - 1$

$y = x - 1 + \ln 2 \qquad$ (A)

normal: $y - \ln 2 = -(x - 1)$

$y = -x + 1 + \ln 2 \qquad$ (B)

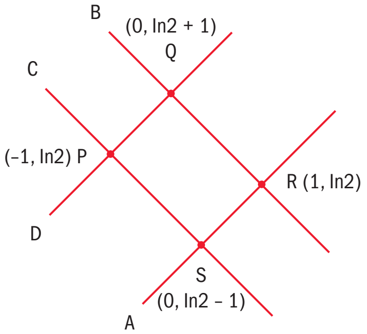
At $(-1, \ln 2)$, $f'(-1) = -1$

tangent: $y - \ln 2 = -(x + 1)$

$y = -x - 1 + \ln 2 \qquad$ (C)

normal: $y - \ln 2 = x + 1$

$y = x + 1 + \ln 2 \qquad$ (D)



Angles are 90° since gradients are ±1

A and C intersection:

$x - 1 + \ln 2 = -x - 1 + \ln 2$

$2x = 0$

$x = 0 \quad (0, \ln 2 - 1)$

B and D intersection:

$-x + 1 + \ln 2 = x + 1 + \ln 2$

$x = 0 \quad (0, \ln 2 + 1)$

$\overrightarrow{PQ} = \begin{pmatrix}1\\1\end{pmatrix}, \overrightarrow{QR} = \begin{pmatrix}1\\-1\end{pmatrix}, \overrightarrow{RS} = \begin{pmatrix}-1\\-1\end{pmatrix}$

$\overrightarrow{SP} = \begin{pmatrix}-1\\1\end{pmatrix}$

$\therefore$ PQ = QR = RS = SP = $\sqrt{2}$

$\therefore$ PQRS is a square, area 2

Exercise 5N

1 **a** arc length $= r\,\theta = \frac{5\pi}{4}$,

sector area $= \frac{1}{2}r^2\,\theta = \frac{25\pi}{8}$

b arc length $= \frac{4 \times 5\pi}{12} = \frac{5\pi}{3}$,

sector area $= \frac{1}{2} \times 16 \times \frac{5\pi}{12} = \frac{10\pi}{3}$

c arc length $= 5.4\,(2\pi - 1.3) \simeq 26.9$ cm

sector area $= \frac{1}{2} \times 5.4^2\,(2\pi - 1.3) \simeq 72.7\ \text{cm}^2$

2 Area = OPQ – OAB $= \frac{1}{2} \times 9^2 \times 0.8 - \frac{1}{2} \times 5^2 \times 0.8$

$= 0.4\,(81 - 25) = 22.4\ \text{m}^2$

Perimeter $= 5 \times 0.8 + 9 \times 0.8 + 4 + 4 = 19.2$ m

3 Length = AB + AC + BC + $3 \times \frac{1}{3}$ perimeter of circle

$= 7.5 + 7.5 + 7.5 + 2\pi \times 7.5$

$\simeq 69.6$ cm

4 **a** Arc length = 5000 stadia

$\therefore\ r \times \frac{7.2\pi}{180} = 5000\ \therefore\ r = \frac{5000 \times 180}{7.2\pi}$ stadia

circumference $= 2\pi r = 2\pi \times \frac{5000 \times 180}{7.2\pi}$ stadia

$= 2 \times \frac{5000 \times 180}{7.2} \times 185\,\text{m}$

$= 46250\,\text{km}$

b Error $= \frac{46250 - 40008}{40008} \times 100\% = 13.5\%$

5 Let AB = x and BC = y, so AC $= \sqrt{x^2 + y^2}$

Crescent APBA + BQCB = semicircle ABA
+ semicircle BCB
– (semicircle ACA – ΔABC)

$= \frac{\pi x^2}{2} + \frac{\pi y^2}{2} - \frac{\pi}{2}(x^2 + y^2) + \Delta\,\text{ABC}$

$= 0 + \Delta\,\text{ABC}$

$= \Delta\,\text{ABC}$

Review exercise

1 $\frac{9^{2n+2} \times 6^{2n-3}}{3^{5n} \times 6 \times 4^{n-2}} = \frac{3^{4n+4} \times 6^{2n-4}}{3^{5n} \times 2^{2n-4}} = \frac{3^{4n+4} \times 3^{2n-4}}{3^{5n}}$

$= \frac{3^{6n}}{3^{5n}} = 3^n$

2 $\frac{8^{\frac{2}{3}} + 4^{\frac{3}{2}}}{16^{\frac{3}{4}}} = \frac{4 + 8}{8} = \frac{3}{2}$

3 **a** $9^x - 12(3^x) + 27 = 0$

$(3^x)^2 - 12(3^x) + 27 = 0$

$(3^x - 9)\,(3^x - 3) = 0$

$3^x = 9$ or $3^x = 3$

$x = 1$ or $x = 2$

b $3^x - \frac{9}{3^x} = 8$

$(3^x)^2 - 8(3^x) - 9 = 0$

$(3^x - 9)\,(3^x + 1) = 0$

$3^x = 9$ or $3^x = -1$

$x = 2$

4 **a** $\log_a x + \log_a 3 - \log_a 7 = \log_a 12$

$\log_a \frac{3x}{7} = \log_a 12$

$3x = 84$

$x = 28$

b $\log_4 x - \log_4 5 = \frac{5}{2}$

$\log_4 \frac{x}{5} = \frac{5}{2}$

$4^{\frac{5}{2}} = \frac{x}{5}$

$\frac{x}{5} = 32 \quad x = 160$

c $\log_3 x - \frac{6}{\log_3 x} = 1$

$(\log_3 x)^2 - \log_3 x - 6 = 0$

$(\log_3 x - 3)\,(\log_3 x + 2) = 0$

$\log_3 x = 3$ or $\log_3 x = -2$

$x = 3^3$ or $x = 3^{-2}$

$x = 27$ or $\frac{1}{9}$

d $\log_7 x + 2\log 7^x = 3$

$\log_7 x + \frac{2}{\log_7 x} = 3$

$(\log_7 x)^2 - 3\log_7 x + 2 = 0$

$(\log_7 x - 1)\,(\log_7 x - 2) = 0$

$\log_7 x = 1$ or 2, so $x = 7$ or 49

5 **a** $xy = 81 \qquad 3\log_x y = 1\ \therefore\ x = y^3$

$y^4 = 81 \qquad \therefore\ y = 3,\ x = 27$

b $y \log_2 4 = x \qquad 2^x + 4^y = 512$

$2y = x \qquad 2^x + 2^{2y} = 512$

$2^x + 2^x = 512$

$2^x = 256$

$\therefore\ x = 8 \quad y = 4$

c $\ln 8 + \ln(x - 6) = 2\ln y \qquad 2y - x = 2$

$\ln 8\,(x - 6) = \ln y^2 \qquad x = 2y - 2$

$8\,(x - 6) = y^2$

$8\,(2y - 8) = y^2$

$y^2 - 16y + 64 = 0$

$(y - 8)^2 = 0 \quad \therefore\ y = 8 \quad x = 14$

6 **a** $\log y + \log \frac{1}{y} = \log 1 = 0$

b $\frac{\log x^5 - \log x^2}{3\log x + \log\sqrt{x}} = \frac{3\log x}{3.5\log x} = \frac{6}{7}$

c $\ln(\ln x^2) - \ln(\ln x) = \ln\left(\frac{\ln x^2}{\ln x}\right) = \ln 2$

7 $\log_2 x + \log_2 x^2 + \log_2 x^3 + \ldots + \log_2 x^m = 3m\,(m + 1)$

$\log_2 x\,(1 + 2 + 3 + \ldots + m) = 3m\,(m + 1)$

$\log_2 x\left(\frac{m}{2}(m+1)\right) = 3m\,(m + 1)$

$\log_2 x = 6$

$x = 2^6 = 64$

8 $y = 5e^{2x} + 8e^{-2x} \qquad \frac{dy}{dx} = 10e^{2x} - 16e^{-2x}$

$\frac{d^2y}{dx^2} = 20e^{2x} + 32e^{-2x} = 4\,(5e^{2x} + 8e^{-2x})$

$= 4y$

9 $y = e^{3x}(2 + 5x)$

$$\frac{dy^2}{dx^2} = e^{3x}(5) + 3e^{3x}(2 + 5x) = e^{3x}(11 + 15\,x)$$

$$\frac{dy^2}{dx^2} = e^{3x}(15) + 3e^{3x}(11 + 15\,x) = e^{3x}(48 + 45\,x)$$

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}(48 + 45x - 6(11 + 15x) + 9(2 + 5x)$$

$$= e^{3x}(48 + 45x - 66 - 90x + 18 + 45x)$$

$$= 0$$

10 $e^x - e^{-x} = 4$

$(e^x)^2 - 4e^x - 1 = 0$

$$e^x = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$$

$e^x > 0 \therefore \quad e^x = 2 + \sqrt{5} \qquad \therefore x = \ln\left(2 + \sqrt{5}\right)$

$$ex + e^{-x} = 2 + \sqrt{5} + \frac{1}{2 + \sqrt{5}}$$

$$= \frac{\left(2 \pm \sqrt{5}\right)^2 + 1}{2 \pm \sqrt{5}} = \frac{4 + 4\sqrt{5} + 5 + 1}{2 + \sqrt{5}}$$

$$= \frac{4\sqrt{5} + 10}{2 + \sqrt{5}} = \frac{2\sqrt{5}\left(2 + \sqrt{5}\right)}{2 + \sqrt{5}}$$

$\sqrt{5}\,e^x + e^{-x} = 2\sqrt{5}$

11 $f(x) = \dfrac{4e^x}{(e^x + 1)^2}$

$$f'(x) = \frac{(e^x + 1)^2 4e^x - 4e^x 2(e^x + 1)e^x}{(e^x + 1)^4}$$

$$= \frac{4e^x(e^x + 1)^2 - 8e^2}{(e^x + 1)^3}$$

$$= \frac{4e^x - 4e^{2x}}{(e^x + 1)^3}$$

$f'(0) = 0 \therefore$ stationary point at (0, 1)

$$f'(x) = \frac{4e^x - 4e^{2x}}{(e^x + 1)^3}$$

$$\Rightarrow f''(x) = \frac{(e^x + 1)^3\left(4e^x - 8e^{2x}\right) - 3\left(e^x + 1\right)^2 e^x\left(4e^x - 4e^{2x}\right)}{\left(e^x + 1\right)^6}$$

$$= \frac{4e^{3x} - 16e^{2x} + 4e^x}{\left(e^x + 1\right)^4}$$

$\therefore f''(0) = \dfrac{4 - 16 + 4}{16} > 0$ so (0, 1) is a maximum point

Points of inflexion: $f''(0) = 0$

$\Rightarrow 4e^{3x} - 16e^{2x} + 4e^x = 0$

$\Rightarrow 4e^{2x} - 16e^x + 4 = 0$

$\Rightarrow e^{2x} - 4e^x + 1 = 0$

$\Rightarrow e^x = \dfrac{4 \pm \sqrt{12}}{2}$

$= 2 \pm \sqrt{3}$

$\Rightarrow x = \ln(2 \pm \sqrt{3})$

12 **a** $f(x) = (\ln x)2 \quad x \in \mathbb{R}, x > 0$

b $f'(x) = \dfrac{2\ln x}{x}$

$$f''(x) = x\left(\frac{2}{x}\right) - 2\ln x = \frac{2 - 2\ln x}{x^2}$$

c $f'(x) = 0 \Rightarrow \ln x = 0 \Rightarrow x = 1 \qquad (1, 0)$

$f''(1) = 2 > 0 \quad \therefore$ minimum at (1, 0)

$f''(x) = 0 \Rightarrow \ln\ x = 1 \Rightarrow x = e \qquad (e, 1)$

x	2	e	3
f″(x)	0.153 > 0	0	−0.0219 < 0

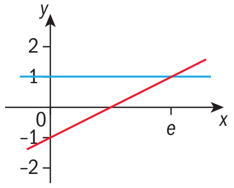
change of sign

$\therefore$ point of inflexion at (e, 1)

d $f'(e) = \frac{2}{e} \qquad y - 1 = \frac{z}{e}(x - e)$

$y = \dfrac{2}{e}x - 1$

e



$1 = \dfrac{2}{e}x - 1$

$2 = \dfrac{2}{e}x$

$\therefore x = e$

area $= \dfrac{1}{2}(2)(e) = e$

13 Radius of sector = AN = $a\sqrt{3}$,

area of sector = $\dfrac{1}{2}$

$$r^2\,\theta = \frac{1}{2}a^2 \times 3 \times \frac{\pi}{3}$$

$$= \frac{\pi a^2}{2}$$

$$\therefore S_1 = \frac{\pi a^2}{2} - \text{area } \Delta\text{ ADE}$$

$$= \frac{\pi a^2}{2} - \frac{1}{2}r^2 \sin 60°$$

$$= \frac{\pi a^2}{2} - \frac{1}{2}3a^2 \times \frac{\sqrt{3}}{2}$$

$$= \frac{\pi a^2}{2} - \frac{3a^2\sqrt{3}}{4} = \frac{a^2\left(2\pi - 3\sqrt{3}\right)}{4}$$

$$\frac{AE}{AC} = \frac{r}{2a} = \frac{\sqrt{3}}{2}, \text{ so } \frac{AM}{AN} = \frac{\sqrt{3}}{2}$$

$$\therefore AM = \frac{\sqrt{3}}{2} \times a\sqrt{3} = \frac{3a}{2}$$

$$\therefore S_2 = \frac{1}{2}AM^2\,\theta - \Delta\text{ AFG}$$

$$= \frac{1}{2} \times \frac{9a^2}{4} \times \frac{\pi}{3} - \frac{1}{2}AM^2 \sin 60°$$

$$= \frac{3\pi a^2}{8} - \frac{1}{2} \times \frac{9a^2}{4} \times \frac{\sqrt{3}}{2}$$

$$= \frac{a^2\left(6\pi - 9\sqrt{3}\right)}{16} = \frac{3a^2\left(2\pi - 3\sqrt{3}\right)}{16}$$

Similarly $S_3 = \frac{1}{2}\text{AC}^2\,\theta - \frac{1}{2}\text{AC}^2 \sin 60°$

$$= \frac{1}{2}\left(\frac{3a\sqrt{3}}{4}\right)^2 \frac{\pi}{2} - \frac{1}{2}\left(\frac{3a\sqrt{3}}{4}\right)^2 \frac{\sqrt{3}}{2}$$

$$\therefore\ S_3 = \frac{1}{2} \times \frac{27a^2}{16} \times \frac{\pi}{3} - \frac{1}{2} \times \frac{27a^2}{16} \times \frac{\sqrt{3}}{2}$$

$$= \frac{a^2}{64}(18\pi - 27\sqrt{3}) = \frac{9a^2}{64}\,(2\pi - 3\sqrt{3})$$

Hence S_1, S_2, S_3 form a geometric series with $r = \frac{3}{4}$.

Total area $= \frac{a}{1-r} = \frac{a^2}{4}\,(2\pi - 3\sqrt{3}) = a^2\,(2\pi - 3\sqrt{3})$

14 Area required = area of regular hexagon of side 8 cm – 6 × area of sector of circle of angle 120° – central circle

$$= 6 \times \text{area of equilateral triangle} - 6 \times \frac{\pi r^2}{3} - \pi r^2\ (r = 4)$$

$$= 6 \times \frac{1}{2} \times 2r \times 2r \sin 60° - 3\pi r^2$$

$$= 6 \times \frac{1}{2} \times 64 \times \frac{\sqrt{3}}{2} - 3\pi \times 16$$

$$= 96\sqrt{3} - 48\pi$$

$$= 15.5\ \text{cm}^2$$

6 Exploring randomness

Answers

Skills check

1 Write in ascending order: 85, 88, 91, 94, 95, 96, 97, 103, 103, 107, 110, 114

- **a** Median is between 96 and 97 = 96.5 kg
- **b** Mode = 103 kg (most frequent)
- **c** Mean = total ÷ number of observations

  $= 1183 \div 12$

  $= 98.6$ kg
- **d** Range = 114 − 85 = 29 kg
- **e** Lower quartile $= \frac{13}{4}$th observation

  $= 91 + \frac{1}{4} \times 3 = 91.75\,\text{kg}$

  Upper quartile $= \frac{3}{4} \times 13$th observation

  $= 9\frac{3}{4}$th observation

  $= 103 + \frac{3}{4} \times 4$

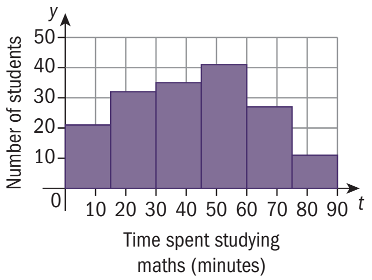
  $= 106$ kg
- **f** IQR = 106 − 91.75 = 14.25 kg

2
- **a** $\binom{8}{3} = \frac{8 \times 7 \times 6}{3!} = 8 \times 7 = 56$
- **b** Total number of ways − number of ways where all 3 have brown eyes

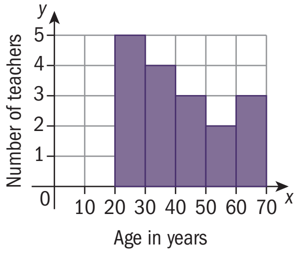
  $= \binom{20}{3} - \binom{12}{3} = 920$

Exercise 6A

1
- **a** Discrete (as they are asked for an answer in whole minutes)
- **b**



2
- **a** Continuous
- **b** 5 + 4 + 3 + 2 + 3 = 17
- **c**



3
- **a** Continuous
- **b**

Mass (kg)	Number of chickens
$1 \le w < 2$	8
$2 \le w < 3$	24
$3 \le w < 4$	50
$4 \le w < 5$	14

- **c** 8 + 24 + 50 + 14 = 96

4
- **a** Continuous
- **b**

Time to get home (mins)	Number of students
$5 \le t < 10$	1
$10 \le t < 15$	2
$15 \le t < 20$	4
$20 \le t < 25$	4
$25 \le t < 30$	2
$30 \le t < 35$	2
$35 \le t < 40$	1
$40 \le t < 45$	1

- **c** 5 mins

5
- **a** First diagram = *D*

  Second diagram = *A*

  Third diagram = *C*

Exercise 6B

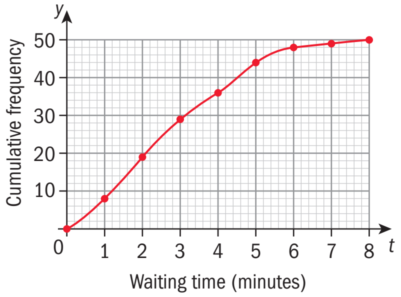
1 **a** 1 goal (highest frequency of 7)

b $170 \le h < 180$ (highest frequency of 10)

2 **a**

t(minutes)	Frequency	CF
$0 \le t < 1$	8	8
$1 \le t < 2$	11	19
$2 \le t < 3$	10	29
$3 \le t < 4$	7	36
$4 \le t < 5$	8	44
$5 \le t < 6$	4	48
$6 \le t < 7$	1	49
$7 \le t < 8$	1	50
	50	

b



% waiting longer than 5 minutes

$$= \frac{6}{50} \times 100\% = 12\%$$

c Estimates from table and graph:

Mean ≈ 2.8 mins, Median ≈ 2.6 mins

Modal interval is $1 \le t < 2$ mins

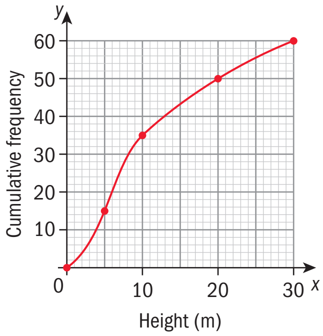
3 **a**

Height (m)	Frequency
$0 \le h < 5$	15
$5 \le h < 10$	20
$10 \le h < 20$	15
$20 \le h < 30$	10
	60

b

Height (m)	CF
5	15
10	35
20	50
30	60

c



c Number less than 18 m = 47

$\therefore \% < 18\text{ m} = \frac{13}{60} \times 100\% \approx 22\%$

d Mean ≈ 11.0 m

Median ≈ 9 m

Modal class is $5 \le h < 10$ m

4 **a** Mode = 3

Median = 3

b Use: $\frac{30+a}{8} = \frac{a+3}{2}$

or $\frac{30+a}{8} = \frac{3+6}{2}$

Both give $a = 6$

This makes the set bimodal at 3 and 6.

5 **a** Series is $\ln a + \frac{1}{2}\ln a + \frac{1}{4}\ln a + \frac{1}{8}\ln a + \ldots$

This is a GP with common ratio $r = \frac{1}{2}$

Since $|r| < |$, this converges with sum

$= \frac{A}{1-r} = \frac{\ln a}{1-\frac{1}{2}} = 2\ln a$

b Mean $= \frac{\sum_{r-1}^{n} 2^{\frac{1}{r-1}} \ln a}{n} = \frac{A\left(1-r^n\right)}{(1-r)n}$

$= \frac{\ln a\left(1-\frac{1}{2^n}\right)}{\frac{1}{2}n} = \frac{2\ln a}{n}\left(1-\frac{1}{2^n}\right)$

c Need $\frac{2\ln a}{n}\left(1-\frac{1}{2^n}\right) < 0.01\ln a$

$\Rightarrow \left(1-\frac{1}{2^n}\right) < 0.005\,n$

$\Rightarrow n = 200$

Exercise 6C

1 Arrange in order: 30, 45, 55, 60, 65, 65, 70, 75, 75, 110, 120, 125

a Range = 125 − 30 = 95 cm

b Median = $6\frac{1}{2}$th reading = $\frac{65 + 70}{2}$ = 67.5 cm

c LQ = $\frac{13}{4}$th reading = $55 + \frac{1}{4} \times 5$ = 56.25 cm

d UQ = $9\frac{3}{4}$th reading = $75 + \frac{3}{4} \times 35$ = 101.25 cm

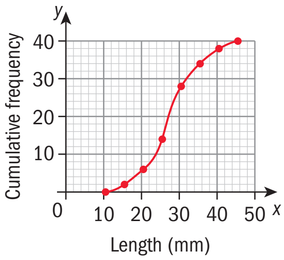
e IQR = 101.25 − 56.25 cm

2 **a** From graph, median = 75 cm

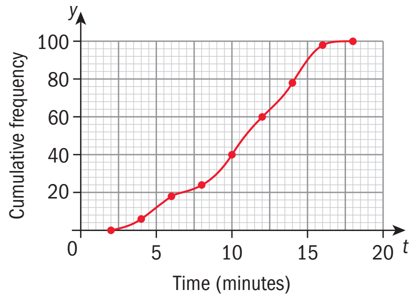
b 77.5 − 72 = 5.5 cm

c 50% of the boxers have a reach with a maximum difference of 5.5 cm

3



4 **a**



i From graph, median = 11 cm

ii IQR = 13.7 – 8.1 = 5.6 mins

b $24 + 36 + p = 92 \Rightarrow p = 32$

$24 + 36 + p + q = 100 \Rightarrow p + q = 40 \Rightarrow q = 8$

5 **a** From graph, number of students = 1100

b Lower quartile = $\frac{4200}{4}$th = 1050th observation ≈ 39

Upper quartile = 3150th observation ≈ 64

Middle 50% lie between 39 and 64

$\Rightarrow a = 39,\ b = 64$

c Number getting more than 80 ≈ 4200 − 3900

= 300

% awarded grade 7 ≈ $\frac{300}{4200} \times 100$ %

= 7.1%

6 **a** 23 mins

b IQR = UQ − LQ = 31 − 16 = 15 mins

c 37 mins

Exercise 6D

1 $\frac{a + b + 15}{6} = 3 \Rightarrow a + b = 3$

$\frac{(a-3)^2 + (b-3)^2 + 1 + 0 + 4 + 4}{6} = \frac{7}{3}$

Solve to find that either $a = 2$, $b = 1$ or $a = 1$, $b = 2$.

Given that $a < b\ \therefore\ a = 1,\ b = 2$.

2 **a** Mean = $\frac{a - 1 + a + a + 2 + a + 3}{4} = \frac{4a + 4}{4} = a + 1$

Variance = $\frac{(-2)^2 + (-1)^2 + 1^2 + 2^2}{4} + \frac{10}{4} = 2.5$

b Mean = $a + 1 + 3 = a + 4$

Variance = 2.5

3 **a** Mean = 9.4

Standard deviation = 1.41

b IQR = 10 − 9 = 1

4 $\frac{2 + 3 + 6 + 9 + x + y}{6} = 6$

$\Rightarrow x + y = 36 - 20 \Rightarrow x + y = 16 \qquad (1)$

$$\frac{(2-6)^2 + (3-6)^2 + (6-6)^2 + (9-6)^2 + (x-6)^2 + (y-6)^2}{6} = 10$$

$\Rightarrow 16 + 9 + 9 + (x-6)^2 + (y-6)^2 = 60$

$\Rightarrow (x-6)^2 + (y-6)^2 = 60 - 34 = 26 \qquad (2)$

From (1), $y = 16 - x$, so

$(x-6)^2 + (10-x)^2 = 26$

$\therefore x^2 - 12x + 36 + 100 - 20x + x^2 = 26$

$\Rightarrow 2x^2 - 32x + 110 = 0$

$\Rightarrow x^2 - 16x + 55 = 0$

$\Rightarrow (x+5)(x-11) = 0$

x is positive, so $x = 11$ and $\therefore\ y = 5$ (from (1))

The last 2 score sums are 5 and 11

∴ score sums are 2, 3, 5, 6, 9, 11

Range = 11 − 2 = 9

IQR = 9.5 − 2.75 = 6.75

5 **a** Mean $= \frac{4k - 2 + k + k + 1 + 2k + 4 + 3k}{5}$

$= \frac{11k + 3}{5}$

b Variance

$= \frac{\sum x^2}{5} - \left(\frac{\sum x}{5}\right)^2$

$= \frac{(4k-2)^2 + k^2 + (k+1)^2 + (2k+4)^2 + 9k^2}{5} - \left(\frac{11k+3}{5}\right)^2$

$= \frac{34}{25}k^2 - \frac{56k^2}{25} + \frac{96}{25}$

c Mean $= \frac{11k+3}{5} - 2 = \frac{11k-7}{5}$

d The variance will be unchanged as the spread of the data about the mean is unaffected.

6 **a** Mean $= \frac{\sum_1^n (2r-1)a}{n} = \frac{a}{n}\left(2\sum_1^n r - n\right)$

$= \frac{a}{n}\left[n(n+1) - n\right] = \frac{a}{n} \times n^2 = an$

Exercise 6E

1 **a** $P(2, 4, 6, 8) = \frac{4}{2}$

b $P(3, 6) = \frac{1}{4}$

c $P(4,8) = \frac{1}{4}$

d $P(1, 2, 3, 5, 6, 7) = \frac{3}{4}$

e $P(1, 2, 3,) = \frac{3}{8}$

2 $\frac{30}{150} = \frac{1}{5}$

3 **a** $P(A) = \frac{1}{2}$

b $P(B) = \frac{4}{6} = \frac{2}{3}$

c $P(A \cup B) = \frac{10}{12} = \frac{5}{6}$

d $P(A \cap B) = \frac{4}{12} = \frac{1}{3}$

e $P(A' \cup B) = \frac{10}{12} = \frac{5}{6}$

5 **a** $P(A) = \frac{27}{36} = \frac{3}{4}$

b $P(B) = \frac{18}{36} = \frac{1}{2}$

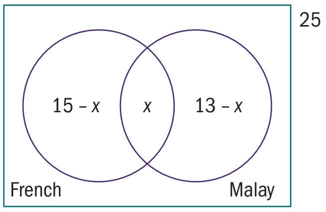
c $P(A\cup B)=\frac{27}{36}=\frac{3}{4}$

d $P(A\cap B)=\frac{18}{36}=\frac{1}{2}$

e $P(A'\cup B')=\frac{18}{36}=\frac{1}{2}$

Exercise 6F

1

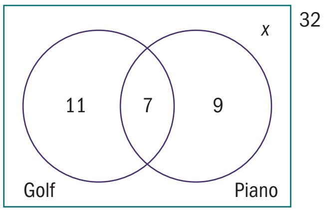


$15-x+x+13-x+5=25$

$\Rightarrow 28-x=20\Rightarrow x=8$

$\therefore$ P (French and Malay) $=\frac{8}{25}$

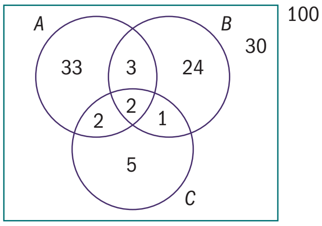
2



a P (golf but not piano) $=\frac{11}{32}$

b P (piano but not golf) $=\frac{9}{32}$

3



a $\frac{33}{100}=0.33$

b $\frac{24}{100}=0.24$

c $\frac{30}{100}=0.3$

4 **a** $P(X\cup Y)=P(X)+P(Y)-P(X\cap Y)$

$=\frac{1}{4}+\frac{1}{8}-\frac{1}{8}$

$=\frac{1}{4}$

b $P(X\cup Y)'=1-P(X\cup Y)=\frac{3}{4}$

5 **a** $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$=0.2+0.5-0.1=0.6$

b $P(A\cup B)'=1-0.6=0.4$

c $P(A'\cup B)=P(A')+P(B)-P(A'\cap B)$

$=0.8+0.5-\left[P(B)-P(A\cap B)\right]$

$=0.5+0.5-0.5+0.1=0.6$

Exercise 6G

1 P (Sophie and Jerome selected) $=\dfrac{\binom{8}{2}}{\binom{10}{4}}$

$=\frac{2}{15}$

2 **a** P (two lines) $=\dfrac{\binom{3}{2}}{\binom{8}{2}}=\frac{3}{8\times7}=\frac{3}{56}$

b P (two different pieces) $=\dfrac{2\times\binom{3}{1}\times\binom{5}{1}}{\binom{8}{2}}=\frac{2\times3\times5}{8\times7}=\frac{15}{28}$

3 **a** $P(R, R, R)=\frac{7}{12}\times\frac{6}{11}\times\frac{5}{10}=\frac{7}{44}$

b P(not all same color)

$=1-P(R,R,R)-P(Y,Y,Y)$

$=1-\frac{7}{44}-\left(\frac{5}{12}\times\frac{4}{11}\times\frac{3}{10}\right)$

$=\frac{35}{44}$

4 **a** P (all orange) $=\dfrac{\binom{5}{3}}{\binom{15}{3}}=\frac{10}{455}=\frac{2}{91}$

b P (all different colors)

$=\dfrac{\binom{4}{1}\times\binom{5}{1}\times\binom{6}{1}}{\binom{15}{3}}=\frac{120}{455}=\frac{24}{91}$

c P (at least one green) = 1 − P (no green)

$=\dfrac{\binom{11}{2}}{\binom{15}{3}}=\frac{55}{455}=\frac{11}{91}$

5 **a** **i** P(Bob scores on 1st shoot)

= P (Bill misses) × P (Bob hits)

$=0.7\times0.25=0.175$

ii P (Bill scores on 3rd shoot) = P (Bill misses, Bob misses, Bill misses, Bob misses, Bill hits)

$=0.7\times0.75\times0.7\times0.75\times0.3\approx0.0827$

iii P (Bill scores on nth shoot) = P (Bill misses, n times, Bob misses, $n-1$ times, Bob hits)

$=(0.7)^n\times(0.75)^{n-1}\times0.25$

b P (Bill wins) $=0.3+0.7\times0.75\times0.3+0.7\times0.75\times0.7\times0.75\times0.3$

$=0.3\left(1+0.525+(0.525)^2+\ldots+3\right)$

$=0.3\times\frac{1}{1-0.525}$

$\therefore p=\frac{0.3}{1-0.525}\Rightarrow p-0.525p=0.3$

$\Rightarrow p=0.3+0.525p$

c $\mathrm{P}\text{ (Bob wins)} = 1 - \mathrm{P}\text{ (Bill wins)}$

$= 1 - p$

$= 1 - \frac{0.3}{1-0.525} \approx 0.368$

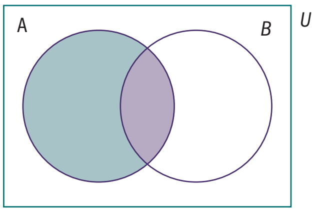
Exercise 6H

1 **a** $\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B)$

$= 0.4 + 0.6 - 0.7$

$= 0.3$

b



$\mathrm{P}(A \cap B') = \mathrm{P}(A) - \mathrm{P}(A \cap B)$

$= 0.4 - 0.3 = 0.1$

c $\mathrm{P}(A' \cup B') = 1 - \mathrm{P}(A \cap B) = 0.7$

2 $\mathrm{P}(160 < h < 180) = \mathrm{P}(h < 180) - \mathrm{P}(h < 160)$

$= 0.75 - 0.2 = 0.55$

3 **a** $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$= 0.6 + 0.55 - 0.2 = 0.95$

b $\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$= 0.55 - 0.2 = 0.35$

c $\mathrm{P}\left[(A \cup B) \setminus (A \cap B)\right] = \mathrm{P}(A' \cap B)$

$+ \mathrm{P}(B' \cap A)$

$= 0.35 + \mathrm{P}(A) - \mathrm{P}(A \cap B)$

$= 0.35 + 0.6 - 0.2 = 0.75$

4 **a** $\mathrm{P}(A) = 0.5 + 0.2 = 0.7$

b $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

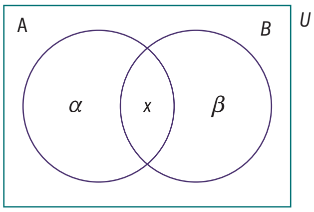
$\therefore 0.85 = 0.7 + \mathrm{P}(B) - 0.2$

$\therefore \mathrm{P}(B) = 0.35$

c $\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$= 0.15$

5



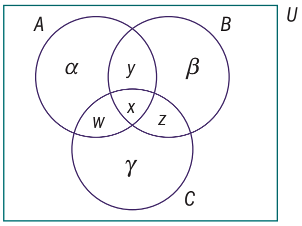
$\mathrm{P}(A) \times \mathrm{P}(B) = (\alpha + x)(\beta + x)$

$= \alpha\beta + x(\alpha + \beta + x)$

But $\alpha\beta >,0$, so $\mathrm{P}(A) \times \mathrm{P}(B) \geq x(\alpha + \beta + x)$

$= \mathrm{P}(A \cap B)\mathrm{P}(A \cup B)$

6



$\mathrm{P}(A \cup B \cup C) = \alpha + \beta + \gamma + x + y + z + w$

$= (\alpha + x + y + w) + (\beta + x + y + z)$

$+ (\gamma + w + x + z) - 2x - y - z - w$

$= \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C)$

$-\left[(x + y) + (w + x) + (x + z) - x\right]$

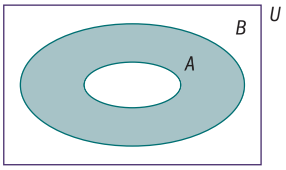
$= \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B)$

$- \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C) + x$

$= \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C)$

$- \mathrm{P}(B \cap C) + \mathrm{P}(A \cap B \cap C)$

7



$\mathrm{P}(B \backslash A) = \mathrm{P}(B \cap A') = \mathrm{P}(B) + \mathrm{P}(A') - \mathrm{P}(B \cap A')$

$= \mathrm{P}(B) + \mathrm{P}(A') - 1 = \mathrm{P}(B) + 1 - \mathrm{P}(A) - 1$

$= \mathrm{P}(B) - \mathrm{P}(A)$

Exercise 6I

1 **a** **i** 0.21

ii $0.19 + 0.14 = 0.33$

b $1200 \times 0.21 = 252$

2 **a** $\frac{27}{100} = 0.27$

b No. If it was fair we would expect around 16 or 17 occurrences of each number. The spinner appears to be biased towards 1.

c $0.15 \times 3000 = 450$

3 **a** $\mathrm{P}(5, 10) = \frac{34 + 68}{100} = \frac{102}{500} = \frac{51}{250}$

b $\mathrm{P}(2, 4, 6, 8, 10, 12) = \frac{6 + 21 + 65 + 63 + 68 + 42}{500} = \frac{265}{500} = \frac{53}{100}$

c $\mathrm{P}(2, 4, 6, 8, 10, 12, 5) = \frac{265 + 34}{500} = \frac{299}{500}$

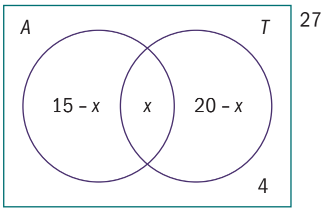
4 **a** $\mathrm{P}(2, 3, 5, 7) = \frac{4}{10} = \frac{2}{5}$

b $\mathrm{P}(2, 3, 5, 7, 4, 8) = \frac{6}{10} = \frac{3}{5}$

c $\mathrm{P}(3, 6, 9, 4, 8) = \frac{5}{10} = \frac{1}{2}$

Exercise 6J

1 **a**



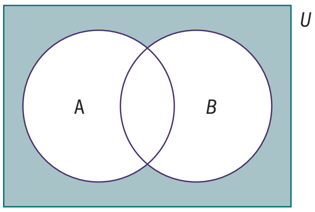
$15 - x + x + 20 - x = 23 \quad \therefore 35 - x = 23 \therefore x = 12$

$\mathrm{P}(T \setminus A) = \frac{20 - x}{27} = \frac{8}{27}$

b $\mathrm{P}(A \cup T) = \frac{3 + 12 + 8}{27} = \frac{23}{27}$

c $\mathrm{P}(T | A) = \frac{\mathrm{P}(T \cap A)}{\mathrm{P}(A)} = \frac{\frac{12}{27}}{\frac{15}{27}} = \frac{12}{15} = \frac{4}{5}$

2 **a**



$\mathrm{P}(A' \cap B') = 1 - \mathrm{P}(A \cup B)$

$\Rightarrow \mathrm{P}(A \cup B) = 1 - 0.35 = 0.65$

$$\begin{aligned}\mathrm{P}(A \cap B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B)\\ &= 0.25 + 0.6 - 0.65\\ &= 0.2\end{aligned}$$

b $\mathrm{P}(A | B) = \mathrm{P}\frac{(A \cap B)}{\mathrm{P}(B)} = \frac{0.2}{0.6} = \frac{1}{3}$

c $\mathrm{P}(B' | A') = \mathrm{P}\frac{(B' \cap A')}{\mathrm{P}(A')} = \frac{0.35}{1 - 0.25} = \frac{0.35}{0.75} = \frac{7}{15}$

3 $\mathrm{P(roller \mid skateboard)} = \frac{\mathrm{P(roller} \cap \mathrm{skateboard)}}{\mathrm{P(skateboard)}}$

$= \frac{0.39}{0.48} = \frac{13}{16}$

4 **a** $\mathrm{P(even \mid not\ mult.\ of\ 4)} = \frac{\mathrm{P(even} \cap \mathrm{not\ mult.\ of\ 4)}}{\mathrm{P(not\ mult.\ of\ 4)}}$

$= \frac{\mathrm{P}(2,22)}{\frac{6}{8}} = \frac{\frac{2}{8}}{\frac{6}{8}} = \frac{1}{3}$

b $\mathrm{P}(<15 | >5)\frac{\mathrm{P}(5 < x < 15)}{\mathrm{P}(x < 15)} = \frac{\frac{2}{8}}{\frac{5}{8}} = \frac{2}{5}$

c $\mathrm{P}(<5 | <15)\frac{\mathrm{P}(x < 5 \text{ and } x < 15)}{\mathrm{P}(x < 15)} = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}$

d $\mathrm{P}(10 < x < 20 | 5 < x < 25)$

$= \frac{\mathrm{P}(10 < x < 20 \text{ and } 5 < x < 25)}{\mathrm{P}(5 < x < 25)}$

$= \frac{\frac{2}{8}}{\frac{4}{8}} = \frac{1}{2}$

5 $\mathrm{P(laptop \mid desktop)} = \frac{\mathrm{P(laptop\ and\ desktop)}}{\mathrm{P(desktop)}} = \frac{0.61}{0.95} = \frac{61}{95}$

6 $\mathrm{P(Spanish \mid Tech)} = \frac{\mathrm{P(Spanish\ and\ Tech)}}{\mathrm{P(Tech)}} = \frac{0.1}{0.6} = \frac{1}{6}$

7 **a** $\mathrm{P}(U \text{ and } V) = \mathrm{P}(U \cap V) = 0$

(mutually exclusive)

b $\mathrm{P}(U | V) = \frac{\mathrm{P}(U \cap V)}{\mathrm{P}(V)} = 0$

c $\mathrm{P}(U \text{ or } V) = \mathrm{P}(U) + \mathrm{P}(V) = 0.63$

8 $\mathrm{P(passed\ 2 \mid passed\ 1)} = \frac{\mathrm{P(1\ and\ 2)}}{\mathrm{P(1)}} = \frac{0.35}{0.52} = 0.673$

$\therefore$ 67.3% of those who passed the first also passed the second.

9 P(white on 2nd | black on 1st)

$= \frac{\mathrm{P(1st\ black\ and\ 2nd\ white)}}{\mathrm{P(black\ on\ 1st)}}$

$= \frac{0.34}{0.47} = \frac{34}{47}$

10 **a** $\mathrm{P(male} \cap \mathrm{left\ handed)} = \frac{5}{50} = \frac{1}{10}$

b $\mathrm{P(right\ handed)} = \frac{43}{50}$

c P(right handed | female)

$= \frac{\mathrm{P(right\ handed\ and\ female)}}{\mathrm{P(female)}} = \frac{\frac{11}{50}}{\frac{13}{50}} = \frac{11}{13}$

11 $\mathrm{P(other\ is\ male \mid one\ is\ male)} = \frac{\mathrm{P(both\ male)}}{\mathrm{P(one\ is\ male)}}$

$= \frac{\frac{1}{4}}{\mathrm{P(not\ 2\ females)}}$

$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

Exercise 6K

1 $\mathrm{P}(A \cap B) = 0.24 = 0.4 \times 0.6 = \mathrm{P}(A) \times \mathrm{P}(B)$

$\therefore A$ and B independent

$\mathrm{P}(B \cap C) = 0.15 \neq \mathrm{P}(B) \times \mathrm{P}(C)$

$\therefore B$ and C not independent

2 $\mathrm{P}(A \cap B) = \mathrm{P(Red\ Queen)} = \frac{2}{52} = \frac{1}{26}$

$\mathrm{P}(A) \times \mathrm{P}(B) = \frac{4}{52} \times \frac{1}{2} = \frac{1}{26}$

$\therefore A$ and B independent

$\mathrm{P}(B \cap C) = \mathrm{P(red\ face\ card)} = \frac{6}{52} = \frac{3}{26}$

$\mathrm{P}(B) \times \mathrm{P}(C) = \frac{1}{2} \times \frac{12}{52} = \frac{6}{52} = \frac{3}{26}$

$\therefore B$ and C are independent

$\mathrm{P}(A \cap C) = \mathrm{P(Queen\ and\ face\ card)}$

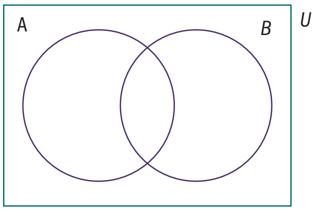
$= \mathrm{P(Queen)} = \frac{4}{52} = \frac{1}{13}$

$\mathrm{P}(A) \times \mathrm{P}(C) = \frac{4}{52} = \frac{12}{52}$

$= \frac{1}{13} \times \frac{3}{13} \neq \frac{1}{13}$

$\therefore$ A and C are not independent

3



a $$\begin{aligned} P(A\cap B') &= P(A) - P(A\cap B) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1-P(B)) \\ &= P(A)P(B') \end{aligned}$$

∴ A and B' are not independent

b $$\begin{aligned} P(A'\cap B) &= P(B) - P(A\cap B) \\ &= P(B) - P(A)P(B) \\ &= P(B)(1-P(A)) \\ &= P(A')P(B) \end{aligned}$$

∴ A' and B are independent

c $$\begin{aligned} &P(A'\cap B') = 1 - P(A\cup B) \\ &= 1 - [P(A) + P(B) - P(A\cap B)] \\ &= 1 - P(A) - P(B) + P(A\cap B) \\ &= (1-P(A))(1-P(B)) \\ &= P(A')P(B') \end{aligned}$$

∴ A' and B' are independent

4 $P(B|A) = \frac{P(B\cap A)}{P(A)} \Rightarrow P(B\cap A) = \frac{3}{4}\times\frac{1}{3} = \frac{1}{4}$

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$\Rightarrow \frac{5}{6} = \frac{1}{3} + P(B) - P(A\cap B)$

$\Rightarrow \frac{5}{6} = \frac{1}{3} + P(B) - \frac{1}{4} \Rightarrow P(B) = \frac{5}{6} - \frac{1}{3} + \frac{1}{4} = \frac{3}{4}$

$P(A)\times P(B) = \frac{1}{3}\times\frac{3}{4} = \frac{1}{4} = P(A\cap B)$

∴ A and B independent

5 **a** $P(A)\times P(B) = P(A\cap B)$

$\Rightarrow P(B) = \frac{P(A\cap B)}{P(A)} = \frac{0.18}{0.45} = \frac{2}{5} = 0.4$

b $$\begin{aligned} P(A\cap B') &= P(A) - P(A\cap B) \\ &= 0.45 - 0.18 = 0.27 \end{aligned}$$

c $$\begin{aligned} P(A'\cap B') &= P(A')\times P(B') \\ &= 0.55\times 0.6 = 0.33 \end{aligned}$$

6 **a** $$\begin{aligned} P(A\cap B) &= P(A) + P(B) - P(A\cup B) \\ &= 3a - \tfrac{5}{8} = P(A)P(B) \end{aligned}$$

$\Rightarrow 2a^2 - 3a + \frac{5}{8} = 0 \Rightarrow 16a^2 - 24a + 5 = 0$

$\Rightarrow (4a-5)(4a-1) = 0 \Rightarrow a = \frac{5}{4}$ or $a = \frac{1}{4}$

Since $P(A)$ and $P(B)$ must be ≤ 1, $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$

7 **a** $P(T, 6) = \frac{1}{2}\times\frac{1}{6} = \frac{1}{12}$

b $P(H, \text{even}) = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$

c $P(H, 3 \text{ or } 6) = \frac{1}{2}\times\frac{1}{3} = \frac{1}{6}$

8 **a** $P(x, x, x, \text{even}) = 1\times 1\times 1\times\frac{1}{2} = \frac{1}{2}$

b $P(x, x, x, \ 0 \text{ or } 5) = 1\times 1\times 1\times\frac{2}{10} = \frac{1}{5}$

c $$\begin{aligned} P(\text{divisible by } 4) &= P(\text{last 2 digits divisible by } 4) \\ &= P(x, x, 0, 0) + P(x, x, 0, 4) \\ &\quad + P(x, x, 0, 8) + \ldots \\ &\quad + P(x, x, 9, 6) \\ &= \frac{25}{100} = \frac{1}{4} \end{aligned}$$

9 Since there is a large number of integers, can assume P(odd) = P(even) = $\frac{1}{2}$

Suppose we select n. Then $P(n \text{ even}) = \left(\frac{1}{2}\right)^n$

∴ P(at least one odd) = $1 - \left(\frac{1}{2}\right)^n$

Need $1 - \left(\frac{1}{2}\right)^n > 0.92$

$\Rightarrow \left(\frac{1}{2}\right)^n < 0.08$

$n \log 0.5 > \log 0.08$

$\Rightarrow n > 3.64$ ∴ need to select 4 integers

10 P(Julia fails to score a winner) = 0.45

∴ P(Julia fails n times in a row) = $(0.45)^n$

∴ P(at least one winner in n shots) = $1 - (0.45)^n$

$1 - (0.45)^n > 0.999$

$\Rightarrow (0.45)^n < 0.001$

$\Rightarrow n > \frac{\log\ 0.001}{\log\ 0.45} \Rightarrow n > 8.65$

∴ Julia needs to hit 9 shots

Exercise 6L

1 P(Rain, not late) = 0.2 × 0.6 = 0.12

2 P(Correct diagnosis) = 0.85 × 0.98 + 0.15 × 0.12
= 0.851

3 P(1 Score out of 2) = 0.75 × 0.15 + 0.25 × 0.8
= 0.3125

4 **a** $$\begin{aligned} P(B') &= \frac{1}{3}\times\frac{2}{5} + \frac{2}{3}\times\frac{1}{2} \\ &= \frac{2}{15} + \frac{1}{3} \\ &= \frac{7}{15} \end{aligned}$$

b $$P(A'\cup B') = \frac{2}{3}\times 1 + \frac{1}{3}\times\frac{2}{5}$$
$$= \frac{2}{3} + \frac{2}{15}$$
$$= \frac{12}{15} + \frac{4}{5}$$

5 **a** $$\text{P(orange, orange, orange)} = \frac{18}{30}\times\frac{17}{29}\times\frac{16}{28}$$
$$= \frac{3}{5}\times\frac{17}{29}\times\frac{4}{7}$$
$$= \frac{204}{1015}$$

b P(at least one purple) = 1 − P(all orange)
$$= 1 - \frac{204}{1015} = \frac{811}{1015}$$

c P(more orange than purple)
$$= P(o, o, o) + P(o, p, o)$$
$$+ P(o, o, p) + P(p, o,\ o)$$
$$= \frac{204}{1015} + \frac{18}{30}\times\frac{12}{29}\times\frac{17}{28} + \frac{18}{30}\times\frac{17}{29}\times\frac{12}{28}$$
$$+ \frac{12}{30}\times\frac{18}{29}\times\frac{17}{28}$$
$$= \frac{204}{1015} + \frac{459}{1015} = \frac{663}{1015}$$

6 **a** $$P(R, R, R) = \frac{26}{52}\times\frac{25}{51}\times\frac{24}{50} = \frac{2}{17}$$

b $$P(H,H,H) = \frac{13}{52}\times\frac{12}{51}\times\frac{11}{50} = \frac{11}{850}$$

c $$\text{P(all same suit)} = \frac{52}{52}\times\frac{12}{51}\times\frac{11}{50} = \frac{22}{425}$$

d $$\text{P(faces in same suit)} = \frac{12}{52}\times\frac{2}{51}\times\frac{1}{50} = \frac{1}{5525}$$

Exercise 6M

1 **a** $$\text{P(even)} = \frac{1}{2}\times\frac{4}{9} + \frac{1}{2}\times\frac{3}{5}$$
$$= \frac{2}{9} + \frac{3}{10} = \frac{47}{90}$$

b $$\text{P(first box | even)} = \frac{P(\text{first box} \cap \text{even})}{P(\text{even})}$$
$$= \frac{\frac{1}{2}\times\frac{4}{9}}{\frac{47}{90}} = \frac{20}{47}$$

2 **a** P(defective) = 0.6 × 0.05 + 0.4 × 0.02 = 0.038

b $$\text{P(first machine | defective)} = \frac{P(f \cap d)}{P(d)}$$
$$= \frac{0.6\times 0.05}{0.038} = \frac{0.03}{0.038}$$
$$= 0.789$$

3 **a** P(V) = 0.6 × 0.35 + 0.4 × 0.75 = 0.51

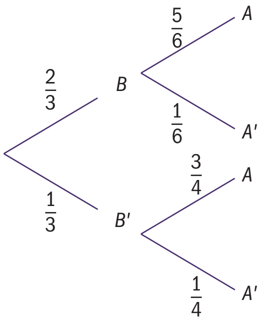
b $$P(V'|G) = 0.25$$

4 **a** $$\text{P(BB from 2nd box)} = \frac{16}{30}\times\frac{13}{20}\times\frac{12}{19} + \frac{14}{30}\times\frac{12}{20}\times\frac{11}{19}$$
$$= \frac{4344}{30\times 20\times 19} = 0.381$$

b P(1st box W | both from 2nd box W)
$$= \frac{P(\text{WWW})}{P(\text{WW 2nd box})}$$

$$= \frac{\frac{14}{30}\times\frac{8}{20}\times\frac{7}{19}}{\frac{16}{30}\times\frac{7}{20}\times\frac{6}{19} + \frac{14}{30}\times\frac{8}{20}\times\frac{7}{19}} = \frac{784}{1456} = \frac{49}{91} = \frac{7}{13}$$

5 **a**



b **i** $$P(A) = \frac{2}{3}\times\frac{5}{6} + \frac{1}{3}\times\frac{3}{4} = \frac{5}{9} + \frac{1}{4}$$
$$= \frac{20+9}{36} = \frac{29}{36}$$

ii $$P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{\frac{2}{3}\times\frac{5}{6}}{\frac{29}{36}} = \frac{20}{29}$$

iii $$P(B'|A') = \frac{P(B'\cap A')}{P(A')} = \frac{\frac{1}{3}\times\frac{1}{4}}{\frac{2}{3}\times\frac{1}{6}\times\frac{1}{3}\times\frac{1}{4}} = \frac{3}{7}$$

6 **a** $$P(6) = \frac{1}{2}\times\frac{2}{3} + \frac{1}{2}\times\frac{1}{6} = \frac{1}{3} + \frac{1}{12} = \frac{5}{12}$$

b $$\text{P(unbiased | not 6)} = \frac{P(\text{unbiased and not 6})}{P(\text{not 6})}$$
$$= \frac{\frac{1}{2}\times\frac{5}{6}}{\frac{1}{2}\times\frac{5}{6} + \frac{1}{2}\times\frac{1}{3}} = \frac{5}{7}$$

7 **a** P(non-smoker) = 0.18 × 0.1 + 0.82 × 0.8
= 0.674

b P(lung problems | heavy smoker)
$$= \frac{P(\text{lung problems and heavy smoker})}{P(\text{heavy smoker})}$$
$$= \frac{0.18\times 0.7}{0.18\times 0.7 + 0.82\times 0.05} = 0.754$$

8 **a** $$P(R) = \frac{1}{3}\times\frac{2}{3} + \frac{1}{3}\times\frac{3}{8} + \frac{1}{3}\times\frac{3}{5}$$
$$= \frac{2}{9} + \frac{1}{8} + \frac{1}{5} = \frac{197}{360}$$

b $$P(C|R) = \frac{P(C\cap R)}{P(R)} = \frac{\frac{1}{3}\times\frac{3}{5}}{\frac{197}{360}} = \frac{1}{5}\times\frac{360}{197} = \frac{72}{197}$$

9 **a** P(on time) = 0.45 × 0.95 + 0.2 × 0.90
+ 0.35 × 0.80
= 0.8875

b $$P(A \mid \text{on time}) = \frac{P(A \text{ and on time})}{P(\text{on time})}$$
$$= \frac{0.45\times 0.95}{0.8875} \approx 0.482$$

c $$P(B \mid \text{late}) = \frac{P(B \text{ and late})}{P(\text{late})} = \frac{0.2\times 0.1}{1-0.8875} \approx 0.178$$

10 **a** $\mathrm{P}(\text{Jar } 2, B) = \frac{5}{15} \times \frac{4}{14} \times \frac{5}{11} + \frac{5}{15} \times \frac{10}{14} \times \frac{6}{11}$

$+ \frac{10}{15} \times \frac{5}{14} \times \frac{6}{11} + \frac{10}{15} \times \frac{9}{14} \times \frac{7}{11}$

$= \frac{1330}{15 \times 14 \times 11} = \frac{19}{33}$

b $\mathrm{P}(\text{Jar 1 = PP} \mid \text{Jar 2 = P}) = \frac{\mathrm{P}(\mathrm{PPP})}{\mathrm{P}(\text{Jar } 2 = \mathrm{P})}$

$= \frac{\frac{5}{15} \times \frac{4}{14} \times \frac{6}{11}}{1 - \mathrm{P}(\mathrm{B})}$

$= \frac{5 \times 4 \times 6}{980} = \frac{6}{49}$

c P(Jar 1 = BB | Jar 2 = P)

$= \frac{\mathrm{P}(\mathrm{BBP})}{\mathrm{P}(\text{Jar } 2 = \mathrm{P})} = \frac{\frac{10}{15} \times \frac{9}{14} \times \frac{4}{11}}{\frac{980}{15 \times 14 \times 11}}$

$= \frac{18}{49}$

11 **a** P(male) = 0.1 × 0.6 + 0.65 × 0.7 + 0.25 × 0.3

$= 0.59$

b $\mathrm{P}(\text{management} \mid \text{male}) = \frac{\mathrm{P}(\text{male and management})}{\mathrm{P}(\text{male})}$

$= \frac{0.1 \times 0.6}{0.59} = 0.102$

c $\mathrm{P}(\text{marketing} \mid \text{female}) = \frac{\mathrm{P}(\text{female and marketing})}{\mathrm{P}(\text{female})}$

$= \frac{0.25 \times 0.7}{1 - 0.59} \approx 0.427$

12 P(second machine | D′)

$= \frac{\mathrm{P}(\text{second machine and D}')}{\mathrm{P}(\mathrm{D}')}$

$= \frac{0.35 \times 0.97}{0.5 \times 0.96 + 0.35 \times 0.97 + 0.15 \times 0.94}$

≈ 0.353

13 P(320, 320 | S = 160)

$= \frac{\mathrm{P}(320, 320, 160)}{\mathrm{P}(S = 160)}$

$$= \frac{\frac{8}{20} \times \frac{7}{19} \times \frac{12}{18}}{\frac{12}{20} \times \frac{11}{19} \times \frac{10}{18} + \frac{12}{20} \times \frac{8}{19} \times \frac{11}{18} + \frac{8}{20} \times \frac{12}{19} \times \frac{11}{18} + \frac{8}{20} \times \frac{7}{19} \times \frac{12}{18}}$$

$= \frac{672}{4104} = \frac{28}{171}$

14 $\mathrm{P}(\mathrm{S}) = \frac{4}{11} \times 0.9 + \frac{4}{11} \times 0.6 + \frac{3}{11} \times 0.2 = 0.6$

15 $\mathrm{P}(\text{vowel}) = \frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} + \frac{3}{7} \times \frac{4}{6} \times \frac{2}{5}$

$+ \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} + \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5}$

$= \frac{90}{7 \times 6 \times 5} = \frac{3}{7}$

Review exercise

1 Mode = 6, so smallest set is 6, 6, x, y

Median = 7, so set is 6, 6, 8, y

Meen = 8, so $\frac{6 + 6 + 8 + y}{4} = 8 \Rightarrow 20 + y = 32$

∴ set is 6, 6, 8, 12

2 A and B independent

$\Rightarrow \mathrm{P}(A \mid B) = \mathrm{P}(B) \Rightarrow \mathrm{P}(B) = \frac{1}{3}$

Let $\mathrm{P}(A) = x$:

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\therefore \frac{11}{12} = x + \frac{1}{3} - x \times \frac{1}{3}$

$\Rightarrow \frac{2}{3}x + \frac{1}{3} = \frac{11}{12}$

$\frac{2}{3}x = \frac{7}{12} \Rightarrow x = \frac{7}{8}$

3 From graph:

a Median = 68 kg

b Middle 50% = 61 − 77 kg

c There are 36 students

4 **a** $\binom{12}{3} = 220$

b Probability = 1 − P (both on committee)

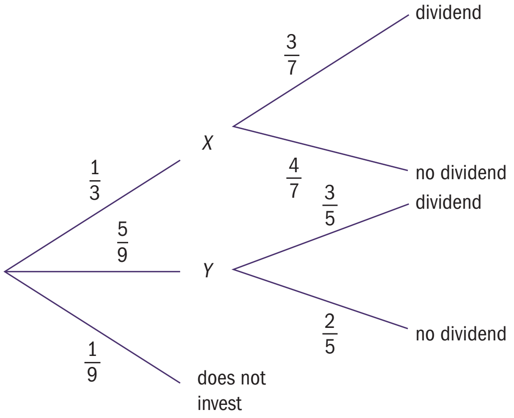
$= 1 - \frac{\binom{10}{1}}{\binom{12}{3}} = 1 - \frac{10}{220} = \frac{11}{22}$

c P(2 girls, 1 boy) + P(3 girls)

$= \frac{\binom{5}{2} \times \binom{7}{1}}{220} + \frac{\binom{5}{3}}{220}$

$= \frac{80}{220} = \frac{4}{11}$

5 **a**



b $\mathrm{P}(\text{dividend}) = \frac{1}{3} \times \frac{3}{7} + \frac{5}{7} \times \frac{3}{5}$

$= \frac{1}{7} + \frac{1}{3} = \frac{10}{21}$

c $\mathrm{P}(Y \mid \text{dividend}) = \frac{\mathrm{P}(Y \text{ and dividend})}{\mathrm{P}(\text{dividend})} = \frac{\frac{5}{9} \times \frac{3}{5}}{\left(\frac{10}{21}\right)}$

$= \frac{7}{10}$

6 $$P(\text{1st G} \mid \text{2nd G}) = \frac{P(GG)}{P(\text{2nd G})}$$

$$= \frac{\frac{5}{12} \times \frac{4}{11}}{\frac{3}{12} \times \frac{5}{11} + \frac{4}{12} \times \frac{5}{11} + \frac{5}{12} \times \frac{4}{11}}$$

$$= \frac{20}{15 + 20 + 20} = \frac{20}{55} = \frac{4}{11}$$

7 **a** $P(\text{prime}) = \frac{6}{36} = \frac{1}{6}$

b $P(\text{even}) = \frac{27}{36} = \frac{3}{4}$

c $P(\text{multiple of } 3) = \frac{20}{36} = \frac{5}{9}$

d $$P(\text{divisible by } 6 \mid \text{even}) = \frac{P(\text{divisible by 6 and even})}{P(\text{even})}$$

$$= \frac{\frac{15}{36}}{\frac{3}{4}} = \frac{5}{9}$$

8 **a** **i** $$\text{mean} = \frac{\sum mi}{n} = \frac{540}{30} = 18$$

ii $$\text{Variance} = \frac{\sum mi^2}{n} - \left(\frac{\sum mi}{n}\right)^2$$

$$= \frac{9990}{30} - 18^2 = 333 - 324 = 9$$

$$\therefore \text{SD} = 3$$

b No, as 95% of the students should get marks within 2 standard deviations of the mean, i.e. between 12 and 24, and 99.7% within 3 standard deviations of the mean, i.e between 9 and 27.

9 **a** $\frac{3n+1}{2}$

b There are n number of the form $3k$ as k runs from 1 to n. But every other one is even, so number of odd numbers is $\frac{n+1}{2}$

Hence $$P(\text{divisible by } 3) = \frac{\frac{n+1}{2}}{\frac{3n+1}{2}} = \frac{n+1}{3n+1}$$

Review exercise

1 $$\text{Mean height} = \frac{23 \times 168 + 17 \times 171 + 8 \times 163 + 20 \times 177}{68}$$

$$= 170.8\text{cm}$$

2 **a** $\frac{10!}{3!3!2!} = 50400$

b P(S_ _ _ _ _ _ _ _ _ _) $= \frac{\frac{9!}{2!3!2!}}{50400} = \frac{15120}{50400} = \frac{3}{10}$

c P(_ _ _ _ _ _ _ _ _ _ consonant)

$$= 1 - \frac{\frac{9!}{3!3!2!}}{50400} - \frac{\frac{9!}{3!3!}}{50400}$$

$$= 1 - \frac{5040}{50400} - \frac{10080}{50400}$$

$$= 1 - \frac{15120}{50400} = \frac{7}{10}$$

3 **a** $\frac{10 \times 10 \times 5}{10 \times 10 \times 10} = \frac{1}{2}$

b $P(\text{abc divisible by } 7) = \frac{142}{1000} = \frac{71}{500}$

c $$P(\text{abc} = \text{perfect square}) = P\left(x^2, 1 \le x \le 31\right)$$

$$= \frac{31}{1000}$$

4 Probaility $= 1 - 0.93 \times 0.93$

$= 0.1351$

5 **a** Mean = 337.5 cm

b Standard deviation = 132.64 km

6

7

8 **a** Mean = 4.69

Standard deviation = 0.552

b

RBC	CF
3.4	0
3.8	7
4.2	22
4.6	58
5.0	80
5.4	107
5.8	120

Median = 60th observation = 4.64

c Number of children with RBC > 5.5 ≈ 120 – 110 = 10

9 **a** $$P(\text{all stats books in first 6 places}) = \frac{\frac{14!}{7!4!3!}}{\frac{20!}{6!7!4!3!}}$$

$$= \frac{14!6!}{20!} = \frac{1}{38760}$$

b $$P(\text{all calculus books together}) = \frac{\frac{14!}{6!4!3!}}{\frac{20!}{6!7!4!3!}}$$

$$= \frac{14!7!}{20!} = \frac{7}{38760}$$

10 **a** P(Keith wins bet i.e. *A* winning)

$$= \frac{4}{11} \times 0.4 + \frac{3}{11} \times 0.55 + \frac{4}{11} \times 0.75$$

$$\approx 0.568$$

b $$P(A \text{ played a higher rank} \mid A \text{ lost}) = \frac{\frac{4}{11} \times 0.6}{1 - 0.568}$$

$$= 0.505$$

7

The evolution of calculus

Answers

Skills check

1 **a** $y = x\ln x \quad \frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$

b $y = \frac{e^{2x-3}}{\sqrt{2-x}}$

$$\frac{dy}{dx} = \frac{\left((2-x)^{\frac{1}{2}}2e^{2x-3} + e^{2x-3}\frac{1}{2}(2-x)^{-\frac{1}{2}}\right)}{(2-x)}$$

$$= \frac{4(2-x)e^{2x-3} + e^{2x-3}}{2(2-x)^{\frac{3}{2}}}$$

$$= \frac{e^{2x-3}(9-4x)}{2(2-x)^{\frac{3}{2}}}$$

c $y = x^4 - \frac{1}{x^4}$

$\frac{dy}{dx} = 4x^3 + 4x^{-5} = 4x^3 + \frac{4}{x^5}$

2 **a** $y = 3x - 2 \quad y^2 = x^2 - 2x + 4$

$x^2 - 2x + 4 = 3x - 2$

$x^2 - 5x + 6 = 0$

$(x - 2)(x - 3) = 0$

$x = 2$ or $3 \quad (2, 4)$ or $(3, 7)$

b $y = 1 - x \quad y = \sqrt{2x+1}$

$1 - x = \sqrt{2x+1} \quad (1)$

$(1 - x)^2 = 2x + 1$

$1 - 2x + x^2 = 2x + 1$

$x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0$ or 4

Check in (1)

if $x = 0$, LHS $= 1$ RHS $= 1$ ✓

if $x = 4$, LHS $= -3$ RHS $= 3$ ✗

$\therefore x = 0 \ (0, 1)$

c $y = \frac{6}{x} + 3x \qquad y = x^3 - 5x$

$\frac{6}{x} + 3x = x^3 - 5x$

$6 + 3x^2 = x^4 - 5x^2$

$x^4 - 8x^2 - 6 = 0$

$x = -2.948$ or 2.948

$(-2.95, -10.9)$ or $(2.95, 10.9)$

3 $s(t) = 3t^4 - t^3 + t$

$v(t) = s'(t) = 12t^3 - 3t^2 + 1$

$a(t) = v'(t) = 36t^2 - 6t$

Exercise 7A

1 $\int -2x\,dx = -x^2 + c$

2 $\int 3x^8\,dx = \frac{x^9}{3} + c$

3 $\int -5x^4\,dx = -x^5 + c$

4 $\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = \frac{-x^{-4}}{4} + c = \frac{-1}{4x^4} + c$

5 $\int \sqrt{x^3}\,dx = \int x^{\frac{3}{2}}\,dx = \frac{2}{5}x^{\frac{5}{2}} + c$

6 $\int \frac{1}{\sqrt{x^3}}\,dx = \int \frac{1}{x^{\frac{3}{2}}}\,dx = \int x^{\frac{-3}{2}}\,dx = -2x^{\frac{-1}{2}} + c = \frac{-2}{\sqrt{x}} + c$

7 $\int \frac{2x}{\sqrt{x}}\,dx = \int 2x^{\frac{1}{2}}\,dx = \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{4}{3}x^{\frac{3}{2}} + c$

8 $\int \frac{-\sqrt[4]{x^5}}{7x^3}\,dx = \int -\frac{1}{7}x^{\frac{-7}{4}}\,dx = -\frac{1}{7}x^{\frac{-3}{4}}\left(\frac{-4}{3}\right) + c$

$$= \frac{4}{21}x^{\frac{-3}{4}} + c$$

Exercise 7B

1 **a** $\int\left(5x^2 - \frac{1}{5x^2}\right)dx = \int\left(5x^2 - \frac{1}{5}x^{-2}\right)dx$

$$= \frac{5x^3}{3} + \frac{1}{5x} + c$$

b $\int (x + 3)(2x - 1)dx = \int (2x^2 + 5x - 3)dx$

$$= \frac{2x^3}{3} + \frac{5x^2}{2} - 3x + c$$

c $\int \frac{x^2 - 1}{x^4}\,dx = \int (x^{-2} - x^{-4})dx$

$$= -x^{-1} + \frac{x^{-3}}{3} + c = -\frac{1}{x} + \frac{1}{3x^3} + c$$

d $\int\left(x + \frac{1}{x}\right)^2 dx = \int (x^2 + 2 + x^{-2})dx$

$$= \frac{x^3}{3} + 2x - \frac{1}{x} + c$$

e $\int \frac{(x+3)(x-4)}{x^5}\,dx = \int (x^{-3} - x^{-4} - 12x^{-5})dx$

$$= \frac{-x^{-2}}{2} + \frac{x^{-3}}{3} + \frac{12x^{-4}}{4} + c$$

$$= -\frac{1}{2x^2} + \frac{1}{3x^3} + \frac{3}{x^4} + c$$

f $\int\left(\sqrt{x}-\frac{5}{\sqrt[3]{x}}\right)\mathrm{d}x=\int(x^{\frac{1}{2}}-5x^{\frac{-1}{3}})\ \mathrm{d}x$

$=\frac{2}{3}x^{\frac{3}{2}}-5x^{\frac{2}{3}}\left(\frac{3}{2}\right)+c$

$=\frac{2}{3}x^{\frac{3}{2}}-\frac{15}{2}x^{\frac{2}{3}}+c$

2 $\frac{\mathrm{d}y}{\mathrm{d}x}=(3x^2-4)\quad(2,-1)$

$y=x^3-4x+c$

$-1=8-8+c\quad\therefore c=-1$

$y=x^3-4x-1$

3 $f'(t)=t+3-\frac{1}{t^2}\qquad\left(1,\frac{-1}{2}\right)$

$f(t)=\frac{t^2}{2}+3t+\frac{1}{t}+c$

$-\frac{1}{2}=\frac{1}{2}+3+1+c\quad\therefore c=-5$

$f(t)=\frac{t^2}{2}+3t+\frac{1}{t}-5$

4 $\frac{\mathrm{d}y}{\mathrm{d}x}=(2x+3)^3=8x^3+3(2x)^2(3)+3(2x)3^2+3^3$

$=8x^3+36x^2+54x+27$

$y=2x^4+12x^3+27x^2+27x+c$

$z=2-12+27-27+c\quad\therefore c=12$

$y=2x^4+12x^3+27x^2+27x+12=\frac{(2x+3)^4+15}{8}$

5 $\frac{\mathrm{d}A}{\mathrm{d}x}=(2x+1)(x^2-1)=2x^3+x^2-2x-1$

$A=\frac{x^4}{2}+\frac{x^3}{3}-x^2-x+c$

$0=\frac{1}{2}+\frac{1}{3}-1-1+c\quad\therefore c=\frac{7}{6}$

$A=\frac{x^4}{2}+\frac{x^3}{3}-x^2-x+\frac{7}{6}$

6 $\frac{\mathrm{d}s}{\mathrm{d}t}=3t-\frac{8}{t^2}$

$s=\frac{3t^2}{2}+\frac{8}{t}+c$

$1.5=1.5+8+c\quad\therefore c=-8$

$s=\frac{3t^2}{2}+\frac{8}{t}-8$

7 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=6x-1\quad\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2-x+c$

$4=12-2+c\quad\therefore c=-6$

$\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2-x-6$

$y=x^3-\frac{x^2}{2}-6x+c$

$0=8-2-12+c\quad\therefore c=6$

$y=x^3-\frac{x^2}{2}-6x+6$

8 $a(t)=6t+1$

$v(t)=3t^2+t+c$

$2=c\quad\therefore v(t)=3t^2+t+2$

$s(t)=t^3+\frac{t^2}{2}+2t+c$

$1=c\quad\therefore s(t)=t^3+\frac{t^2}{2}+2t+1$

Exercise 7C

1 $\int(3x-1)^7\ \mathrm{d}x=\frac{(3x-1)^8}{24}+c$

2 $\int-2\sqrt{2x+1}\ \mathrm{d}x=\int-2(2x+1)^{\frac{1}{2}}\ \mathrm{d}x$

$=\frac{-2(2x+1)^{\frac{3}{2}}}{2\left(\frac{3}{2}\right)}+c=\frac{-2(2x+1)^{\frac{3}{2}}}{3}+c$

3 $\int\frac{1}{(4x-1)^5}\ \mathrm{d}x=\int(4x-1)^{-5}\ \mathrm{d}x$

$=\frac{(4x-1)^{-4}}{4(-4)}+c=\frac{-1}{16(4x-1)^4}+c$

4 $\int\frac{2}{\sqrt[4]{3-x}}\ \mathrm{d}x=\int2(3-x)^{-\frac{1}{4}}\ \mathrm{d}x$

$=\frac{2(3-x)^{\frac{3}{4}}}{\frac{-3}{4}}+c=\frac{-8(3-x)^{\frac{3}{4}}}{3}+c$

5 $\int\left(\frac{2}{(2-5x)^{\frac{1}{3}}}+\sqrt[3]{1-x}\right)\mathrm{d}x=\int\left(2(2-5x)^{\frac{-1}{3}}+(1-x)^{\frac{1}{3}}\right)\mathrm{d}x$

$=\frac{2(2-5x)^{\frac{2}{3}}}{-5\left(\frac{2}{3}\right)}+\frac{(1-x)^{\frac{4}{3}}}{\frac{-4}{3}}+c$

$=\frac{-3(2-5x)^{\frac{2}{3}}}{5}-\frac{3(1-x)^{\frac{4}{3}}}{4}+c$

6 $\int\left(4\sqrt{2-3x}-6(3x+2)^{\frac{2}{3}}\right)\mathrm{d}x$

$=\int\left(4(2-3x)^{\frac{1}{2}}-6(3x+2)^{\frac{2}{3}}\right)\mathrm{d}x$

$=\frac{4(2-3x)^{\frac{3}{2}}}{-3\left(\frac{3}{2}\right)}-\frac{6(3x+2)^{\frac{5}{3}}}{3\left(\frac{5}{3}\right)}+c$

$=\frac{-8(2-3x)^{\frac{3}{2}}}{9}-\frac{6(3x+2)^{\frac{5}{3}}}{5}+c$

Exercise 7D

1 $\int-5\mathrm{e}^{-2x}\ \mathrm{d}x=\frac{5\mathrm{e}^{-2x}}{2}+c$

2 $\int\frac{1}{\mathrm{e}^{3x+2}}\ \mathrm{d}x=\int\mathrm{e}^{-3x-2}\ \mathrm{d}x$

$=\frac{-1}{3}\mathrm{e}^{-3x-2}+c$

3 $$\int\left(\sqrt[3]{e^x}-\frac{2}{e\sqrt{e^{2x}}}\right)dx=\int\left(e^{\frac{x}{3}}-2e^{-x-1}\right)dx$$
$$=3e^{\frac{x}{3}}+2e^{-x-1}+c$$

4 $$\int 3^x\,dx=\frac{3^x}{\ln 3}+c$$

5 $$\int\frac{1}{3^{2x}}\;dx=\int 3^{-2x}\;dx=-\frac{3^{-2x}}{2\ln 3}+c$$

6 $$\int 4^{1-x}\,dx=-\frac{4^{1-x}}{\ln 4}+c$$

7 $\int m^{ax+b}\,dx$ Let $u = ax + b$

$$\frac{du}{dx}=a \quad \therefore dx=\frac{du}{a}$$

$$\int m^{ax+b}\,dx=\int m^u\frac{du}{a}=\frac{1}{a}\int m^u\,du$$
$$=\frac{1}{a}\;\frac{m^u}{\ln(m)}+c$$
$$=\frac{1}{a\ln(m)}\;m^{ax+b}+c$$

Exercise 7E

1 $$\int\frac{1}{3x}\,dx=\frac{1}{3}\ln|x|+c$$

2 $$\int-\frac{6}{x}dx=-6\ln|x|+c$$

3 $$\int\frac{1}{2-3x}\,dx=-\frac{1}{3}\ln|2-3x|+c$$

4 $$\int\frac{5}{3-5x}\,dx=-\ln|3-5x|+c$$

5 $$\int-2(4+3x)^{-1}\,dx=-\frac{2}{3}\ln|4+3x|+c$$

Exercise 7F

1 $$\int_1^3\left(3x+\frac{1}{x^2}\right)dx=\left[\frac{3x^2}{2}-\frac{1}{x}\right]_1^3=\frac{27}{2}-\frac{1}{3}-\frac{3}{2}+1$$
$$=12-\frac{1}{3}+1$$
$$=\frac{38}{3}$$

2 $$\int_0^2 3\sqrt{4x+1}\,dx=3=\left[\frac{(4x+1)^{\frac{3}{2}}}{4\left(\frac{3}{2}\right)}\right]_0^2$$
$$=\frac{1}{2}\left[(4x+1)^{\frac{3}{2}}\right]_0^2$$
$$=\frac{1}{2}(27-1)=13$$

3 $$\int_{-1}^2-2e^{1-3x}\,dx=\frac{2}{3}\left[e^{1-3x}\right]_{-1}^2$$
$$=\frac{2}{3}\left(e^{-5}-e^4\right)=\frac{2\left(1-e^9\right)}{3e^5}$$

4 $$\int_1^3 3.2^{x+1}dx=\frac{3}{\ln 2}\left[2^{x+1}\right]_1^3$$
$$=\frac{3}{\ln 2}(16-4)=\frac{36}{\ln 2}$$

5 $$\int_{-2}^0 2(1-3x)^5dx=\frac{2}{-3(6)}\;\left[(1-3x)^6\right]_{-2}^0$$
$$=-\frac{1}{9}\;\left[1-7^6\right]$$
$$=13072$$

6 $$\int_1^4\frac{1-\sqrt{x}}{\sqrt{x}}dx=\int_1^4\left(x^{-\frac{1}{2}}-1\right)dx$$
$$=\left[2x^{\frac{1}{2}}-x\right]_1^4$$
$$=(4-4)-(2-1)=-1$$

Exercise 7G

1 $$\int_{-1}^0(2r-1)^4dx=\left[\frac{(2r-1)^5}{10}\right]_{-1}^0$$
$$=\frac{1}{10}\left((-1)-(-3)^5\right)=\frac{242}{10}=\frac{121}{5}$$

2 Not possible, $s \neq 0$

3 Not possible, $x \neq \pm 1,\ 1 \in [0, 2]$

4 $$\int_0^1\frac{dx}{(2x+1)^3}=\int_0^1(2x+1)^{-3}dx$$
$$=-\frac{1}{4}\left[(2x+1)^{-2}\right]_0^1=-\frac{1}{4}\left[\frac{1}{(2x+1)^2}\right]_0^1$$
$$=-\frac{1}{4}\left(\frac{1}{9}-1\right)=\frac{2}{9}$$

5 Not possible, $x \neq -1$

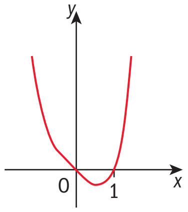
6 $$\int_0^1\left(\frac{3}{3x+4}-\frac{2}{x+1}\right)dx=\left[\ln|3x+4|-2\ln|x+1|\right]_0^1$$
$$=(\ln7-2\ln2)-(\ln4-2\ln1)$$
$$=\ln7-\ln4-\ln4$$
$$=\ln\frac{7}{16}$$

7 $$\int_{-1}^1\frac{e^x+4}{e^x}dx=\int_{-1}^1\left(1+4e^{-x}\right)dx$$
$$=\left[x-4e^{-x}\right]_{-1}^1$$
$$=\left(1-4e^{-1}\right)-(-1-4e)$$
$$=2-\frac{4}{e}+4e$$

8 $$\int_0^2 10^x dx=\frac{1}{\ln 10}\left[10^x\right]_0^2$$
$$=\frac{1}{\ln 10}(100-1)$$
$$=\frac{99}{\ln 10}$$

Exercise 7H

1 $y = x^4 - x = x(x^3 - 1)$



$$\int_{-1}^{0}(x^4 - x)\mathrm{d}x = \left[\frac{x^5}{5}-\frac{x^2}{2}\right]_{-1}^{0}$$

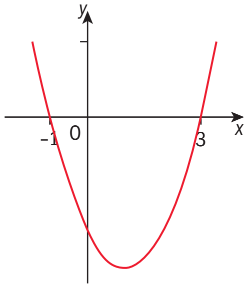
$$= 0-\left(\frac{-1}{5}-\frac{1}{2}\right)=\frac{7}{10}$$

$$\int_{0}^{1}(x^4 - x)\ \mathrm{d}x = \left[\frac{x^5}{5}-\frac{x^2}{2}\right]_{0}^{1}$$

$$=\frac{1}{5}-\frac{1}{2}=-\frac{3}{10}$$

∴ area $=\frac{7}{10}+\frac{3}{10}$ = 1 sq. unit

2 $y = x^2 - 2x - 3 = (x - 3)(x + 1)$



$$A = \int_{-3}^{-1}(x^2 - 2x - 3)\mathrm{d}x$$

$$=\left[\frac{x^3}{3}-x^2-3x\right]_{-3}^{-1}$$

$$=\left(\frac{-1}{3}-1+3\right)-(-9-9+9)$$

$$=\frac{32}{3}\text{ sq. units}$$

3 $\displaystyle\int_{-1}^{1}(x^2 - 2x - 3)\ \mathrm{d}x = \left[\frac{x^3}{3}-x^2-3x\right]_{-1}^{1}$

$$=\left(\frac{1}{3}-1-3\right)-\left(\frac{-1}{3}-1+3\right)=\frac{-16}{3}$$

∴ required area = $\frac{32}{3}+\frac{16}{3}$ = 16 sq. units

4 $\displaystyle\int_{\ln 3}^{3}\left(\mathrm{e}^x-3\right)\mathrm{d}x=\left[\mathrm{e}^x-3x\right]_{\ln 3}^{3}$

$$=\mathrm{e}^3-9-3+3\ln 3$$

$$=\mathrm{e}^3+3\ln 3-12$$

$$\int_{0}^{\ln 3}\left(\mathrm{e}^x-3\right)\mathrm{d}x=\left[\mathrm{e}^x-3x\right]_{0}^{\ln 3}$$

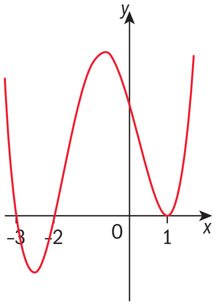
$$=3-3\ln 3-1$$

$$=-(3\ln 3-2)$$

∴ area = $\mathrm{e}^3 + 3\ln 3 - 12 + 3\ln 3 - 2$

= $\mathrm{e}^3 + 6\ln 3 - 14$

5 $y = x^4 + 3x^3 - 3x^2 - 7x + 6$



$$\int_{-3}^{-2}(x^4 + 3x^3 - 3x^2 - 7x + 6)\mathrm{d}x$$

$$=\left[\frac{x^5}{5}+\frac{3x^4}{4}-x^3-\frac{7x^2}{2}+6x\right]_{-3}^{-2}$$

$$=\left(-\frac{32}{5}+12+8-14-12\right)$$

$$-\left(-\frac{243}{5}+\frac{243}{4}+27-\frac{63}{2}-18\right)$$

$$= -12.4 - (-10.35) = -2.05$$

$$\int_{-2}^{1}(x^4 + 3x^3 - 3x^2 - 7x + 6)\mathrm{d}x$$

$$=\left[\frac{x^5}{5}+\frac{3x^4}{4}-x^3-\frac{7x^2}{2}+6x\right]_{-2}^{1}$$

$$=\left(\frac{1}{5}+\frac{3}{4}-1-\frac{7}{2}+6\right)-(-12.4) = 14.85$$

area = 2.05 + 14.85 = 16.9 sq. units

6 $y=\sqrt{4-x}\quad x = 0,\quad x = 4$

$$A = \int_{0}^{4}(4-x)^{\frac{1}{2}}\ \mathrm{d}x=\left[-\frac{2}{3}(4-x)^{\frac{3}{2}}\right]_{0}^{4}$$

$$=-\frac{2}{3}\left(0-4^{\frac{3}{2}}\right)=\frac{16}{3}\text{ sq. units}$$

7 $y=\frac{1}{x^2}+1\quad x=\frac{1}{2},\ x = 5$

$$A = \int_{\frac{1}{2}}^{5}(x^{-2} + 1)\ \mathrm{d}x=\left[\frac{-1}{x}+x\right]_{\frac{1}{2}}^{5}$$

$$=\left(\frac{-1}{5}+5\right)-\left(-2+\frac{1}{2}\right)$$

= 6.3 sq. units

8 $y = 2^x\quad x = 1,\ x = 2$

$$A=\int_{1}^{2}2^x\ \mathrm{d}x=\frac{1}{\ln 2}\left[2^x\right]_1^2$$

$$=\frac{1}{\ln 2}(4-2)=\frac{2}{\ln 2}\text{ sq. units}$$

9 $y = 2\mathrm{e}^{-x+1} - 1\quad x = 0,\ x = 3$

$$A = \int_{0}^{3}|2\mathrm{e}^{-x+1} - 1|\,\mathrm{d}x = 3.32\text{ sq. units}$$

10 $y=\frac{1}{x+2}\quad x = -1,\ x = 2$

$$A=\int_{-1}^{2}\frac{1}{x+2}\ \mathrm{d}x=\left[\mathrm{In}\,|\,x+2\,|\right]_{-1}^{2}$$

= ln4 − ln1 = ln4 = 2ln2 sq. units

11 $y = \frac{2}{3-4x} \quad x = 1, x = 3$

$$\int_1^3 \frac{2}{3-4x}\,dx = \frac{2}{-4}\left[\text{In}\,|3-4x|\right]_1^3$$

$$= -\frac{1}{2}(\ln 9 - \ln 1) = -\frac{1}{2}\ln 9 = -\ln 3$$

$\therefore$ area = ln3 sq. units

12 $y = -x^3 + 6x^2 + x - 30 \quad x\text{-intercepts: } -2, 3, 5$

$$\int_{-2}^3 (-x^3 + 6x^2 + x - 30)dx$$

$$= \left[\frac{-x^4}{4} + 2x^3 + \frac{x^2}{2} - 30x\right]_{-2}^3$$

$$= \left(\frac{-81}{4} + 54 + \frac{9}{2} - 90\right) - (-4 - 16 + 2 + 60)$$

$$= -51.75 - 42$$

$$= -93.75$$

$$\int_3^5 (-x^3 + 6x^2 + x - 30)dx = \left[\frac{-x^4}{4} + 2x^3 + \frac{x^2}{2} - 30x\right]_3^5$$

$$= \left(\frac{-625}{4} + 250 + \frac{25}{2} - 150\right) - (-51.75)$$

$$= -43.75 + 51.75 = 8$$

$\therefore$ area = 93.75 + 8 = 101.75 sq. units

13 $y = \begin{cases} x^2 & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \end{cases}$

$$\text{Area} = \int_0^1 x^2 dx + \int_1^2 (2 - x)dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{3} + (4 - 2) - \left(2 - \frac{1}{2}\right) = \frac{5}{6} \text{ sq. units}$$

14 $y = \begin{cases} \sqrt{x} & 0 \le x \le 1 \\ x^2 & 1 \le x \le 2 \end{cases}$

$$\text{Area} = \int_0^1 x^{\frac{1}{2}} dx + \int_1^2 x^2\,dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 + \left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{2}{3} + \frac{8}{3} - \frac{1}{3}$$

$$= 3 \text{ sq. units}$$

Exercise 7I

1 $y = x^2 + 1 \quad y = 1, \quad y = 10$

$$x = \sqrt{y-1} \quad A = \int_1^{10} (y-1)^{\frac{1}{2}}\,dy = \left[\frac{2}{3}(y-1)^{\frac{3}{2}}\right]_1^{10} = \frac{2}{3}(9)^{\frac{3}{2}}$$

$= 18$ sq. units

2 $y = \sqrt{x} \quad y = 0, \quad y = 4$

$x = y^2$

$$A = \int_0^4 y^2\,dy = \left[\frac{y^3}{3}\right]_0^4 = \frac{64}{3} \text{ sq. units}$$

3 $y = \sqrt{4-x} \quad y = 0, \quad y = 2$

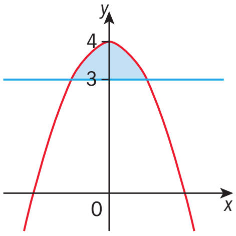
$y^2 = 4 - x$

$x = 4 - y^2$

$$A = \int_0^2 (4 - y^2)\,dy = \left[4y - \frac{y^3}{3}\right]_0^2$$

$$= 8 - \frac{8}{3} = \frac{16}{3} \text{ sq.units}$$

4 $y = 4 - x^2 \quad y = 3, \quad y = 4$



$x^2 = 4 - y$

$x = (4-y)^{\frac{1}{2}}$

$$A = 2\int_3^4 (4-y)^{\frac{1}{2}}dy = 2\left[\frac{-2}{3}(4-y)^{\frac{3}{2}}\right]_3^4$$

$$= \frac{-4}{3}(0-1) = \frac{4}{3} \text{ sq.units}$$

5 $y = \frac{1}{\sqrt{-x+4}} \quad y = \frac{1}{2} \quad y = 2$

$-x + 4 = \frac{1}{y^2} \quad x = 4 - \frac{1}{y^2}$

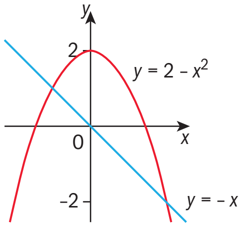
$$A = \int_{\frac{1}{2}}^2 \left(4 - \frac{1}{y^2}\right)dy = \left[4y + \frac{1}{y}\right]_{\frac{1}{2}}^2$$

$$= \left(8 + \frac{1}{2}\right) - (2 + 2)$$

$$= 4\frac{1}{2} \text{ sq. units}$$

Exercise 7J

1



$2 - x^2 = -x$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or -1

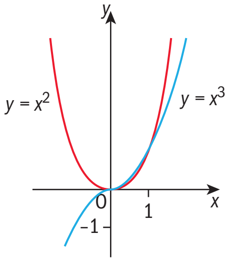
$$\text{Area} = \int_{-1}^2 (2 - x^2 + x)\,dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^2$$

$$= \left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{9}{2} \text{ sq. units}$$

2



$x^3 = x^2$

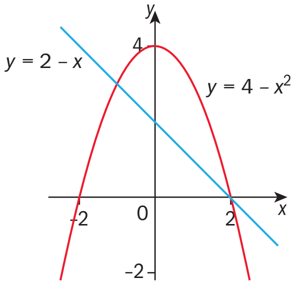
$x^2\,(x-1) = 0$

$x = 0 \text{ or } 1$

$$A = \int_0^1 (x^2 - x^3)\mathrm{d}x = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \text{ sq. units}$$

3



$4 - x^2 = 2 - x$

$x^2 - x - 2 = 0$

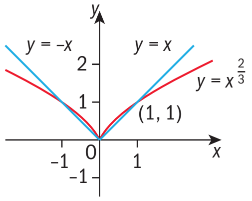
$(x-2)\,(x+1) = 0$

$x = -1 \text{ or } 2$

$$\text{Area} = \int_{-1}^{2} (4 - x^2 - (2 - x))\,\mathrm{d}x = \int_{-1}^{2} (2 - x^2 + x)\mathrm{d}x$$

$$= \frac{9}{2} \text{ sq. units} \quad \text{(see qn. 1)}$$

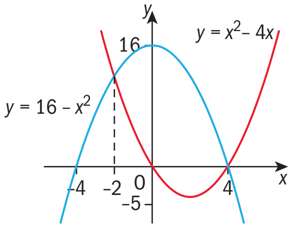
4



$$\text{Area} = 2\int_0^1 \left(x^{\frac{2}{3}} - x\right)\mathrm{d}x = 2\left[\frac{3}{5}x^{\frac{5}{3}} - \frac{x^2}{2}\right]_0^1$$

$$= 2\left(\frac{3}{5} - \frac{1}{2}\right) = \frac{1}{5}$$

5



$16 - x^2 = x^2 - 4x$

$2x^2 - 4x - 16 = 0$

$x^2 - 2x - 8 = 0$

$(x-4)\,(x+2) = 0$

$x = 4 \text{ or } -2$

$$A = \int_{-2}^{4} (16 - x^2 - (x^2 - 4x))\,\mathrm{d}x$$

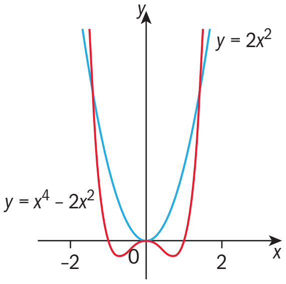
$$= \int_{-2}^{4} (16 - 2x^2 + 4x)\,\mathrm{d}x$$

$$= \left[16x - \frac{2}{3}x^3 + 2x^2\right]_{-2}^{4}$$

$$= \left(64 - \frac{128}{3} + 32\right) - \left(-32 + \frac{16}{3} + 8\right)$$

$= 72 \text{ sq. units}$

6



$x^4 - 2x^2 = 2x^2$

$x^4 - 4x^2 = 0$

$x^2\,(x^2 - 4) = 0$

$x^2\,(x-2)\,(x+2) = 0$

$x = 0, \pm 2$

$$A = \int_{-2}^{2} (2x^2 - (x^4 - 2x^2))\,\mathrm{d}x = \int_{-2}^{2} (4x^2 - x^4)\mathrm{d}x$$

$$= \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2} = \left(\frac{32}{3} - \frac{32}{5}\right) - \left(\frac{-32}{3} + \frac{32}{5}\right)$$

$$= \frac{128}{15} \text{ sq. units}$$

7 $2x^3 + 5x^2 + x - 2 = 8 - 4x - 20x^2 - 8x^3$

$10x^3 + 25x^2 + 5x - 10 = 0$

$2x^3 + 5x^2 + x - 2 = 0$

$x = -2, -1, \frac{1}{2}$

$$\text{Area} = \int_{-2}^{-1} (10x^3 + 25x^2 + 5x - 10)\mathrm{d}x$$

$$+ \int_{-1}^{\frac{1}{2}} (-10x^3 - 25x^2 - 5x + 10)\mathrm{d}x$$

$$= \left[\frac{5x^4}{2} + \frac{25x^3}{3} + \frac{5x^2}{2} - 10x\right]_{-2}^{-1}$$

$$+ \left[\frac{-5x^4}{2} - \frac{25x^3}{3} - \frac{5x^2}{2} + 10x\right]_{-1}^{\frac{1}{2}}$$

$$= \left(\frac{5}{2} - \frac{25}{3} + \frac{5}{2} + 10\right) - \left(40 - \frac{200}{3} + 10 + 20\right)$$

$$+ \left(-\frac{5}{32} - \frac{25}{24} - \frac{5}{8} + 5\right) - \left(-\frac{5}{2} + \frac{25}{3} - \frac{5}{2} - 10\right)$$

$$= \frac{20}{3} - \frac{10}{3} + \frac{305}{96} + \frac{20}{3} = \frac{1265}{96}$$

$= 13.2 \text{ sq. units (3 sf)}$

8 $x^4 - 4 = \frac{1}{1+x}$ $(x > 0)$

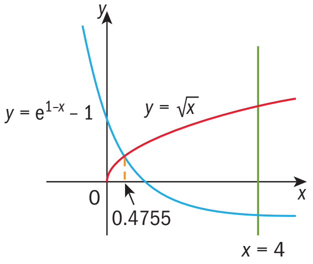
$(x^4 - 4)(1 + x) = 1$

$x^5 + x^4 - 4x - 5 = 0$

$x = 1.449$

$A = \int_0^{1.449} \left(\frac{1}{1+x} - x^4 + 4\right) dx = 5.41$ sq. units

9



$A = \int_{0.4755}^{4} (\sqrt{x} - e^{1-x} + 1)\, dx = 7.00$ sq. units

10 $y = x^2$ $x = \sqrt{y}$

$\int_a^4 y^{\frac{1}{2}} dy = \int_0^a y^{\frac{1}{2}} dy$

$\left[\frac{2}{3} y^{\frac{3}{2}}\right]_a^4 = \left[\frac{2}{3} y^{\frac{3}{2}}\right]_0^a$

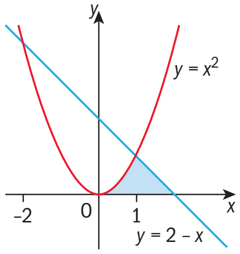
$4^{\frac{3}{2}} - a^{\frac{3}{2}} = a^{\frac{3}{2}}$

$2a^{\frac{3}{2}} = 8$

$a^{\frac{3}{2}} = 4$

$a = 4^{\frac{2}{3}}$

11



$x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

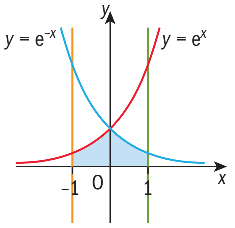
$x = -2$ or 1

$A = \int_0^1 x^2\, dx + \int_1^2 (2 - x)\, dx$

$A = \left[\frac{x^3}{3}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 = \frac{1}{3} + (4 - 2) - \left(2 - \frac{1}{2}\right)$

$= \frac{5}{6}$ sq. units

12

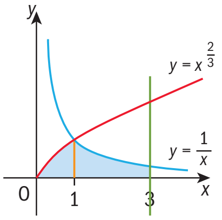


$\text{Area} = 2\int_0^1 e^{-x}\, dx$

$= 2\left[-e^{-x}\right]_0^1$

$= 2(-e^{-1} + 1) = 2\left(1 - \frac{1}{e}\right)$ or 1.26 sq. units

13



$A = \int_0^1 x^{\frac{2}{3}} dx + \int_1^3 \frac{1}{x} dx$

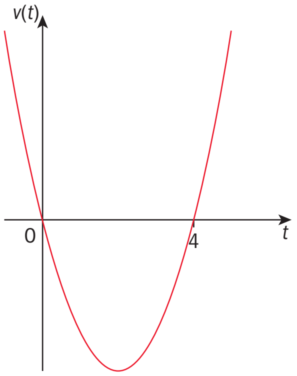
$= \left[\frac{3}{5} x^{\frac{5}{3}}\right]_0^1 + \left[\ln|x|\right]_1^3$

$= \frac{3}{5}(1 - 0) + \ln 3 - \ln 1$

$= \frac{3}{5} + \ln 3$ sq. units

Exercise 7K

1 $v(t) = t(t - 4)$

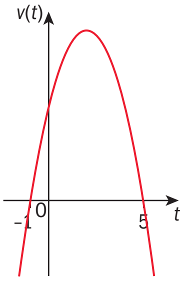


$\text{distance} = \left|\int_0^4 (t^2 - 4t)\, dt\right|$

$= \left|\left[\frac{t^3}{3} - 2t^2\right]_0^4\right|$

$= \left|\frac{64}{3} - 32\right| = \left|\frac{-32}{3}\right| = \frac{32}{3}$ m

2 $v(t) = 5 + 4t - t^2 = (1 + t)(5 - t)$



a $\text{distance} = \int_0^1 (5 + 4t - t^2) dt$

$= \left[5t + 2t^2 - \frac{t^3}{3}\right]_0^1$

$= 5 + 2 - \frac{1}{3} = \frac{20}{3}$ m

b distance = $\int_1^5 (5 + 4t - t^2)\,dt$

$+ \left|\int_5^6 (5 + 4t - t^2)\,dt\right|$

$$= \left[5t + 2t^2 - \frac{t^3}{3}\right]_1^5 + \left|\left[5t + 2t^2 - \frac{t^3}{3}\right]_5^6\right|$$

$$= \left(25 + 50 - \frac{125}{3}\right) - \frac{20}{3} + \left|(30 + 72 - 72) - \left(25 + 50 - \frac{125}{3}\right)\right|$$

$$= \frac{80}{3} + \left|\frac{-10}{3}\right| = 30\,\text{m}$$

3 $a(t) = 1 - e^{-2t} \quad 0 \le t \le 3$

$v(t) = t + \frac{1}{2}e^{-2t} + c$

$v(0) = 0 \quad \therefore 0 = \frac{1}{2} + c \quad \therefore c = -\frac{1}{2}$

$v(t) = t + \frac{1}{2}e^{-2t} - \frac{1}{2}$

distance = $\int_0^3 \left(t + \frac{1}{2}e^{-2t} - \frac{1}{2}\right)dt$

$$= \left[\frac{t^2}{2} - \frac{1}{4}e^{-2t} - \frac{1}{2}t\right]_0^3 = 3.25\,\text{m}$$

4 $v(t) = 10 + 5\,e^{-0.5t}$

a $a(t) = -2.5e^{-0.5t} < 0 \;\therefore$ always negative

b distance = $\int_0^2 (10 + 5e^{-0.5t})\,dt$

$= 26.3\,\text{m}$

Exercise 7L

1 $y = (x - 1)^2 - 1 = x^2 - 2x$

$v = \pi\int_0^1 (x^2 - 2x)^2\,dx = \pi\int_0^1 (x^4 - 4x^3 + 4x^2)\,dx$

$$= \pi\left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3}\right]_0^1$$

$= \frac{8}{15}\pi$ cu. units or 1.68 cu. units

2 $y = 1 + \sqrt{x}$

$v = \pi\int_0^2 (1 + \sqrt{x})^2\,dx = \pi\,^2\,(1 + 2\sqrt{x} + x)\,dx$

$$= \pi\left[x + \frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2}\right]_0^2$$

$$= \pi\left(2 + \tfrac{4}{3}(2\sqrt{2}) + 2\right) = \pi\left(4 + \tfrac{4}{3}(2\sqrt{2})\right)$$

$= \frac{4\pi}{3}(3 + 2\sqrt{2})$ cu. units or 24.4 cu. units

3 $y = \frac{x^2}{2} \quad x^2 = 2y$

$v = \pi\int_0^2 2y\,dy = \pi\left[y^2\right]_0^2 = 4\pi$ cu. units

4 $y = \sqrt{2x - x^2} \quad y^2 = 2x - x^2$

$v = \pi\int_1^2 (2x - x^2)\,dx = \pi\left[x^2 - \frac{x^3}{3}\right]_1^2$

$= \pi\left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = \frac{2\pi}{3}$ cu. units

5 $y = x^{\frac{3}{2}} \quad x = y^{\frac{2}{3}} \quad x^2 = y^{\frac{4}{3}}$

$v = \pi\int_1^3 y^{\frac{4}{3}}\,dy = \pi\left[\frac{3}{7}y^{\frac{7}{3}}\right]_1^3$

$= \frac{3\pi}{7}(3^{\frac{7}{3}} - 1) = 16.1$ cu. units

6 $y = \frac{x}{12}\sqrt{36 - x^2} \quad y^2 = \frac{x^2}{144}(36 - x^2) = \frac{x^2}{4} - \frac{x^4}{144}$

$v = \pi\int_0^6 \left(\frac{x^2}{4} - \frac{x^4}{144}\right)dx = \pi\left[\frac{x^3}{12} - \frac{x^5}{720}\right]_0^6$

$= \pi(18 - 10.8)$

$= 7.2\pi$ cu. units

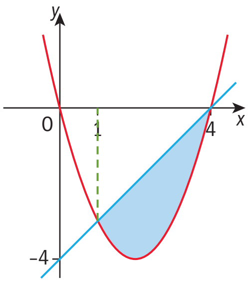
Exercise 7M

1 $y = x \quad y = \frac{x}{2}$

$v = \pi\int_2^5 \left(x^2 - \frac{x^2}{4}\right)dx = \pi\int_2^5 \frac{3x^2}{4}\,dx$

$= \pi\left[\frac{x^3}{4}\right]_2^5 = \pi\left(\frac{125}{4} - 2\right) = \frac{117\pi}{4}$ cu. units

2



$x - 4 = x^2 - 4x$

$x^2 - 5x + 4 = 0$

$(x - 1)(x - 4) = 0$

$x = 1$ or 4

$v = \pi\int_1^4 ((x^2 - 4x)^2 - (x - 4)^2)\,dx$

$= \pi\int_1^4 (x^4 - 8x^3 + 16x^2 - x^2 + 8x - 16)\,dx$

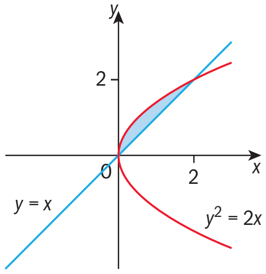
$= \pi\int_1^4 (x^4 - 8x^3 + 15x^2 + 8x - 16)\,dx$

$$= \pi\left[\frac{x^5}{5} - 2x^4 + 5x^3 + 4x^2 - 16x\right]_1^4$$

$$= \pi\left[\left(\frac{1024}{5} - 512 + 320 + 64 - 64\right) - \left(\frac{1}{5} - 2 + 5 + 4 - 16\right)\right]$$

$= \frac{108\pi}{5}$ cu. units

3



$x^2 = 2x \quad x(x-2) = 0$

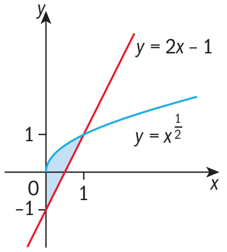
$x = 0 \text{ or } 2 \quad y = 0 \text{ or } 2$

$$v = \pi\int_0^2 \left(y^2 - \left(\frac{y^2}{2}\right)^2\right) \mathrm{d}y$$

$$v = \pi\int_0^2 \left(y^2 - \frac{y^4}{16}\right)\mathrm{d}y$$

$$v = \pi\left[\frac{y^3}{3} - \frac{y^5}{20}\right]_0^2 = \pi\left(\frac{8}{3} - \frac{32}{20}\right) = \frac{16}{15}\pi \text{ cu. units}$$

4



$y = 2x - 1 \quad y = x^{\frac{1}{2}}$

$x = \frac{y+1}{2} \quad\quad x = y^2$

$x^2 = \frac{(y+1)^2}{4} \quad x^2 = y^4$

$$v = \pi\int_{-1}^{1} \frac{(y+1)^2}{4}\,\mathrm{d}y - \pi\int_0^1 y^4\,\mathrm{d}y$$

$$= \pi\left[\frac{(y+1)^3}{12}\right]_{-1}^{1} - \pi\left[\frac{y^5}{5}\right]_0^1$$

$$= \pi\left(\frac{8}{12}\right) - \pi\left(\frac{1}{5}\right) = \frac{7\pi}{15} \text{ cu. units}$$

Review exercise

1 $\frac{\mathrm{d}y}{\mathrm{d}x} = ax + \frac{b}{x^2}$ $(-1, 2)$ $(-2, 0)$ = stationary point

when $x = -2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\therefore -2a + \frac{b}{4} = 0$

$\therefore b = 8a$ (1)

$y = \frac{ax^2}{2} - \frac{b}{x} + c$

$(-2, 0)$ $0 = 2a + \frac{b}{2} + c$ (2)

$(-1, 2)$ $2 = \frac{a}{2} + b + c$ (3)

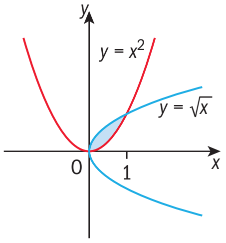
(2) − (3) $-2 = \frac{3a}{2} - \frac{b}{2}$ $b = 8a$

$\therefore -2 = \frac{3a}{2} - 4a \Rightarrow -4 = 3a - 8a \Rightarrow -4 = -5a$

$a = \frac{4}{5} \quad b = \frac{32}{5} \quad c = -2a - \frac{b}{2} = -\frac{8}{5} - \frac{16}{5} = \frac{-24}{5}$

$y = \frac{2}{5}x^2 - \frac{32}{5x} - \frac{24}{5}$

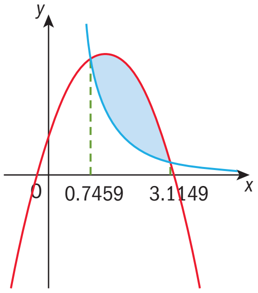
2



$$\text{Area} = \int_0^1 (x^{\frac{1}{2}} - x^2)\mathrm{d}x$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq. units}$$

3



$$v = \pi\int_{0.7459}^{3.1149} \left((1+3x-x^2)^2 - \left(\frac{2}{x}\right)^2\right)\mathrm{d}x$$

$= 41.3$ cu. units

4 **a** $\int_1^2 \left(x + \frac{1}{x^2} - \frac{1}{x^4}\right)\mathrm{d}x = \int_1^2 (x + x^{-2} - x^{-4})\,\mathrm{d}x$

$$= \left[\frac{x^2}{2} - x^{-1} + \frac{x^{-3}}{3}\right]_1^2 = \left[\frac{x^2}{2} - \frac{1}{x} + \frac{1}{3x^3}\right]_1^2$$

$$= \left(2 - \frac{1}{2} + \frac{1}{24}\right) - \left(\frac{1}{2} - 1 + \frac{1}{3}\right) = \frac{41}{24}$$

b $\int_1^4 \frac{5x^4 - 4}{\sqrt{x}}\,\mathrm{d}x = \int_0^4 (5x^{\frac{3}{2}} - 4x^{-\frac{1}{2}})\mathrm{d}x$

$$= \left[2x^{\frac{5}{2}} - 8x^{\frac{1}{2}}\right]_1^4 = (2(4)^{\frac{5}{2}} - 8(4)^{\frac{1}{2}}) - (2 - 8)$$

$= (64 - 16) - (-6) = 54$

c $\int_1^2 \frac{1}{x-3}\mathrm{d}x = [\ln|x-3|]_1^2 = \ln 1 - \ln 2 = -\ln 2$

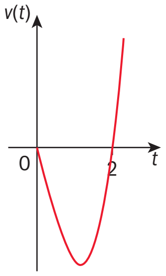
d $\int_1^{\mathrm{e}} \frac{1}{1-4x}\mathrm{d}x = -\frac{1}{4}[\ln|1-4x|]_1^{\mathrm{e}}$

$= -\frac{1}{4}(\ln|1 - 4\mathrm{e}| - \ln 3)$

$= -\frac{1}{4}(\ln(4\mathrm{e} - 1) - \ln 3) = -\frac{1}{4}\ln\left(\frac{4\mathrm{e}-1}{3}\right)$

Review exercise

1 $v(t) = t^3 - 4t = t(t^2 - 4) = t(t - 2)(t + 2)$



$$\int_0^2 (t^3 - 4t)\mathrm{d}t = \left[\frac{t^4}{4} + 2t^2\right]_0^2$$

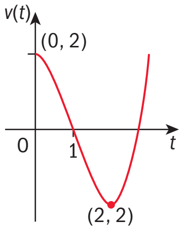
$$= 4 - 8 = -4$$

$$\int_2^3 (t^3 - 4t)\mathrm{d}t = \left[\frac{t^4}{4} - 2t^2\right]_2^3$$

$$= \left(\frac{81}{4} - 18\right) - (-4) = \frac{25}{4}$$

$\therefore$ total distance $= 4 + \frac{25}{4} = \frac{41}{4}$ m

2 $v(t) = t^3 - 3t^2 + 2$



$$\int_0^1 (t^3 - 3t^2 + 2)\,\mathrm{d}t = \left[\frac{t^4}{4} - t^3 + 2t\right]_0^1$$

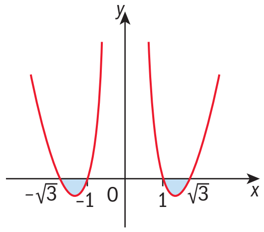
$$= \frac{1}{4} - 1 + 2 = \frac{5}{4}$$

$$\int_1^2 (t^3 - 3t^2 + 2)\,\mathrm{d}t = \left[\frac{t^4}{4} - t^3 + 2t\right]_1^2$$

$$= (4 - 8 + 4) - \left(\frac{5}{4}\right) = \frac{-5}{4}$$

$\therefore$ total distance $= \frac{5}{4} + \frac{5}{4} = \frac{5}{2}$ m

3



$y = x^2 - 4 + \frac{3}{x^2}$

$x^2 - 4 + \frac{3}{x^2} = 0$

$x^4 - 4x^2 + 3 = 0$

$(x^2 - 1)(x^2 - 3) = 0 \quad x = \pm 1, \pm\sqrt{3}$

$$\int_1^{\sqrt{3}} \left(x^2 - 4 + \frac{3}{x^2}\right)\mathrm{d}x = \left[\frac{x^3}{3} - 4x - \frac{3}{x}\right]_1^{\sqrt{3}}$$

$$= (\sqrt{3} - 4\sqrt{3} - \sqrt{3}) - \left(\frac{1}{3} - 4 - 3\right) = -4\sqrt{3} + \frac{20}{3}$$

$\therefore$ total area $= 2\left(4\sqrt{3} - \frac{20}{3}\right) = 8\sqrt{3} - \frac{40}{3}$ sq. units

4 **a** $\displaystyle\int \frac{3x^4 + 6}{x^2}\,\mathrm{d}x = \int 3x^2 + \frac{6}{x^2}\,\mathrm{d}x = x^3 - \frac{6}{x} + c$

b $\displaystyle\int \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)\mathrm{d}x = \int \left(x^2 - \frac{1}{x^2}\right)\mathrm{d}x = \frac{x^3}{3} + \frac{1}{x} + c$

c $\displaystyle\int \frac{1}{2 - 3x}\,\mathrm{d}x = \frac{-1}{3}\ln|2 - 3x| + c$

d $\displaystyle\int \frac{2}{\sqrt{1-4x}}\,\mathrm{d}x = \int 2(1-4x)^{\frac{-1}{2}}\,\mathrm{d}x = \frac{2}{-4}\,\frac{(1-4x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$$= -\sqrt{1-4x} + c$$

e $\displaystyle\int \left(2\mathrm{e}^{-3x} + \sqrt[3]{\mathrm{e}^x}\right)\mathrm{d}x = \int \left(2\mathrm{e}^{-3x} + \mathrm{e}^{\frac{x}{3}}\right)\mathrm{d}x$

$$= \frac{-2}{3}\mathrm{e}^{-3x} + 3\mathrm{e}^{\frac{x}{3}} + c = \frac{-2}{3}\mathrm{e}^{-3x} + 3\sqrt[3]{\mathrm{e}^x} + c$$

5
$$\begin{array}{r} x+2 \\ 2x-1\overline{)2x^2+3x} \\ \underline{2x^2-x} \\ 4x \\ \underline{4x-2} \\ 2 \end{array}$$

$$\therefore \frac{2x^2 + 3x}{2x - 1} = x + 2 + \frac{2}{2x - 1}$$

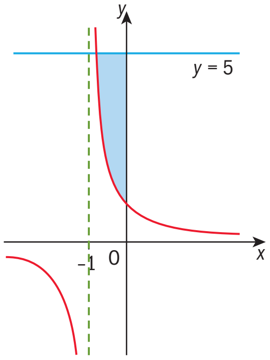
$$\int_1^2 \frac{2x^2 + 3x}{2x - 1}\,\mathrm{d}x = \int_1^2 \left(x + 2 + \frac{2}{2x - 1}\right)\mathrm{d}x$$

$$= \left[\frac{x^2}{2} + 2x + \ln|2x - 1|\right]_1^2$$

$$= (2 + 4 + \ln 3) - \left(\frac{1}{2} + 2 + \ln 1\right)$$

$$= \frac{7}{2} + \ln 3$$

6



$y = \frac{1}{x+1} \quad \therefore x + 1 = \frac{1}{y}$

$x = \frac{1}{y} - 1$

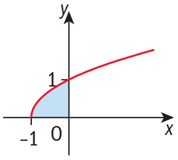
$$\int_1^5 \left(\frac{1}{y} - 1\right) dy = [\ln y - y]_1^5$$

$$= (\ln 5 - 5) - (\ln 1 - 1)$$

$$= \ln 5 - 4$$

$\therefore$ area = 4 − ln5 sq. units

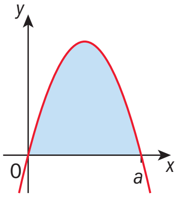
7



$$A = \int_{-1}^{0} (x+1)^{\frac{1}{2}}\, dx = \left[\frac{2}{3}(x+1)^{\frac{3}{2}}\right]_{-1}^{0}$$

$= \frac{2}{3}$ sq. units

8



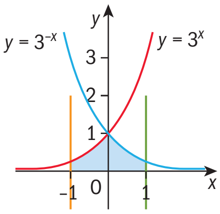
$$\int_0^a (3ax - 3x^2)\, dx = 4$$

$$\left[\frac{3ax^2}{2} - x^3\right]_0^a = 4$$

$$\frac{3a^3}{2} - a^3 = 4$$

$$\frac{a^3}{2} = 4 \quad a^3 = 8 \quad \therefore a = 2$$

9



$$v = 2\pi \int_0^1 (3^{-x})^2\, dx = 2\pi \int_0^1 3^{-2x}\, dx$$

$$= \frac{2\pi}{-2\ln 3}\left[3^{-2x}\right]_0^1$$

$$= -\frac{\pi}{\ln 3}(3^{-2} - 1)$$

$$= \frac{\pi}{\ln 3}\left(\frac{8}{9}\right)$$

$= \frac{8\pi}{9\ln 3}$ cu. units

8 Ancient mathematics and modern methods

Answers

Skills check

1 Using Δ's CFE and CBA, $\frac{\text{EF}}{6.5} = \frac{\text{CF}}{\text{CB}}$

Using Δ's CDF and BAF, $\frac{4}{\text{CF}} = \frac{6.5}{\text{BF}}$

$$\frac{4}{\text{CF}} = \frac{6.5}{\text{CB} - \text{CF}}$$

$$\frac{\text{CB} - \text{CF}}{\text{CF}} = \frac{6.5}{4} \Rightarrow \frac{\text{CB}}{\text{CF}} - 1 = 1.625 \Rightarrow \frac{\text{CB}}{\text{CF}} = 2.265$$

$$\therefore \frac{\text{EF}}{6.5} = \frac{1}{2.625} \quad \therefore \text{EF} = \frac{51}{21} = 2.48 \text{ m}$$

Exercise 8A

1 **a** $\text{B}\hat{\text{A}}\text{C} = 90^\circ - 28^\circ = 62^\circ$

$\sin 28^\circ = \frac{\text{AB}}{8} \quad \therefore \text{AB} = 8\sin 28^\circ = 3.76 \text{ cm}$

$\cos 28^\circ = \frac{\text{BC}}{8} \quad \therefore \text{BC} = 8\cos 28^\circ = 7.06 \text{ cm}$

b $\text{QR}^2 = 7^2 - 4.2^2 \quad \therefore \text{QR} = \sqrt{31.36} = 5.6 \text{ cm}$

$\sin \text{P}\hat{\text{R}}\text{Q} = \frac{4.2}{7} \quad \therefore \text{P}\hat{\text{R}}\text{Q} = 36.9^\circ$

$\text{Q}\hat{\text{P}}\text{R} = 90^\circ - 36.9^\circ = 53.1^\circ$

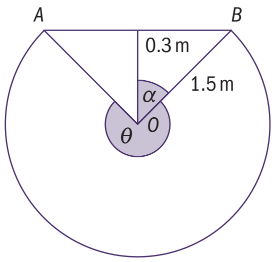
2 $\tan\alpha = \frac{\text{BT}}{\text{AB}} \quad \therefore \text{BT} = 30\tan 52.3^\circ = 38.8 \text{ m}$

3 $\cos\theta = \frac{r}{15} \quad \therefore r = 15\cos 1.23 = 5.013\ldots$

arc length $= 2\pi r = 31.50\ldots$

$\therefore 15\varphi = 31.50\ldots \quad \therefore \varphi = 2.10 \text{ radians}$

4



$\cos\alpha = \frac{0.3}{1.5} \quad \therefore \alpha = 1.369\ldots$

$2\alpha = 2.738\ldots$

$\theta = 2\pi - 2.738\ldots = 3.544$

$\text{Area } \Delta\text{AOB} = \frac{1}{2}(1.5)(1.5)\sin 2.738\ldots$

$= 0.4409\ldots$

$\text{Area major sector AOB} = \frac{1}{2}1.5^2(3.544\ldots) = 3.987\ldots$

Cross-sectional area of milk $= 0.4409\ldots + 3.987\ldots$

$= 4.428\ldots$

$\therefore$ Volume of milk $= 4.428\ldots \times 3 = 13.3 \text{ m}^3$

Exercise 8B

1 **a** $\sin 144^\circ = \sin 36^\circ$

b $\cos 210^\circ = -\cos 30^\circ$

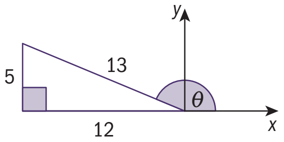
c $\tan 230^\circ = \tan 50^\circ$

d $\sin\left(\frac{7\pi}{8}\right) = \sin\left(\frac{\pi}{8}\right)$

e $\tan\left(\frac{7\pi}{3}\right) = \tan\frac{\pi}{3}$

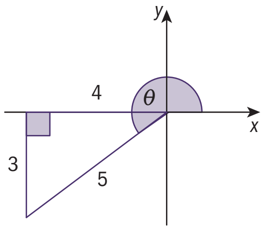
f $\cos\left(\frac{7\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right)$

2



$\cos\theta = -\frac{12}{13} \quad \tan\theta = -\frac{5}{12}$

3



$\sec\theta = -\frac{5}{4} \Rightarrow \cos\theta = -\frac{4}{5}$

$\tan\theta = \frac{3}{4} \quad \sin\theta = -\frac{3}{5}$

4 **a** $2 + 4\cos\theta$

i max value = 6 when $\theta = 2\pi$

ii min value = –2 when $\theta = \pi$

b $5 - 3\sin\theta$

i max value = 8 when $\theta = \frac{3\pi}{2}$

ii min value = 2 when $\theta = \frac{\pi}{2}$

c $2\sin\theta - 1$

i max value = 1 when $\theta = \frac{\pi}{2}$

ii min value = −3 when $\theta = \frac{3\pi}{2}$

d $-2\cos\theta - 3$

i max value = −1 when $\theta = \pi$

ii min value = −5 when $\theta = 2\pi$

Investigation – trigonometric identities

1 **a** $\sin\theta = \cos(90-\theta), \quad \cos\theta = \sin(90-\theta),$

$\tan\theta = \frac{1}{\tan(90-\theta)}$

b $\tan\theta = \frac{\sin\theta}{\cos\theta}$

c $\sin^2\theta + \cos^2\theta = 1$

e $\tan^2\theta + 1 = \sec^2\theta$

f $\cot^2\theta + 1 = \csc^2\theta$

Investigation – exact values of sin, cos and tan

1 **a** $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \tan\frac{\pi}{4} = 1$

b $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \cos\frac{\pi}{3} = \frac{1}{2} \quad \tan\frac{\pi}{3} = \sqrt{3}$

$\sin\frac{\pi}{6} = \frac{1}{2} \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

Exercise 8C

1 $\sin\theta = \frac{1}{4} \quad \frac{\pi}{2} \le \theta \le \pi$

$\cos^2\theta = 1-\sin^2\theta = 1-\frac{1}{16} = \frac{15}{16} \quad \therefore \cos\theta = \frac{-\sqrt{15}}{4}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-1}{\sqrt{15}}$

2 $\cos\theta = \frac{-12}{13} \quad 0 \le \theta \le \pi \quad$ (θ lies in 2nd quadrant)

$\sin^2\theta = 1-\cos^2\theta = 1-\frac{144}{169} = \frac{25}{169} \quad \therefore \sin\theta = \frac{5}{13}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{5}{12}$

3 $\sin\left(\arcsin\left(\frac{\sqrt{3}}{2}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right) = \sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$

$= \sin\frac{\pi}{6} = \frac{1}{2} \quad \text{(QED)}$

4 $\frac{\sin\theta}{1-\cos\theta} + \frac{1-\cos\theta}{\sin\theta} = \frac{\sin^2\theta + (1-\cos\theta)^2}{\sin\theta(1-\cos\theta)}$

$= \frac{\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}{\sin\theta(1-\cos\theta)}$

$= \frac{2-2\cos\theta}{\sin\theta(1-\cos\theta)} = \frac{2(1-\cos\theta)}{\sin\theta(1-\cos\theta)} = \frac{2}{\sin\theta} \quad \text{(QED)}$

5 $\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$

$= \frac{1}{\cos\theta\sin\theta} = \sec\theta\csc\theta \quad \text{(QED)}$

$(\sin\theta + \cos\theta)(\tan\theta + \cot\theta) = (\sin\theta + \cos\theta)\sec\theta\csc\theta$

$= \sin\theta\sec\theta\csc\theta + \cos\theta\sec\theta\csc\theta$

$= \sec\theta + \csc\theta \quad \text{(QED)}$

6 $\cot^2\theta - \cos^2\theta = \frac{\cos^2\theta}{\sin^2\theta} - \cos^2\theta = \frac{\cos^2\theta - \cos^2\theta\sin^2\theta}{\sin^2\theta}$

$= \frac{\cos^2\theta(1-\sin^2\theta)}{\sin^2\theta} = \frac{\cos^2\theta\cos^2\theta}{\sin^2\theta}$

$= \cos^4\theta\csc^2\theta \quad \text{(QED)}$

Exercise 8D

1 **a** $\sin 75^\circ = \sin(30^\circ + 45^\circ)$

$= \sin 30^\circ\cos 45^\circ + \cos 30^\circ \sin 45^\circ$

$= \frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$

$= \frac{1+\sqrt{3}}{2\sqrt{2}}$

b $\tan 15^\circ = \tan(60^\circ - 45^\circ) = \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ}$

$= \frac{\sqrt{3}-1}{1+\sqrt{3}}$

c $\sec 105^\circ = \frac{1}{\cos(60^\circ + 45^\circ)} = \frac{1}{\cos 60^\circ\cos 45^\circ - \sin 60^\circ \sin 45^\circ}$

$= \frac{1}{\frac{1}{2}\times\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}} = \frac{2\sqrt{2}}{1-\sqrt{3}} = -\frac{2\sqrt{2}}{\sqrt{3}-1}$

2 **a** $\cos 70^\circ\cos 10^\circ + \sin 70^\circ \sin 10^\circ = \cos 60^\circ = \frac{1}{2}$

b $\frac{\tan 75^\circ}{\tan 15^\circ} = \frac{\tan(30^\circ + 45^\circ)}{\tan(45^\circ - 30^\circ)} = \frac{\frac{\tan 30^\circ + \tan 45^\circ}{1-\tan 30^\circ \tan 45^\circ}}{\frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ}}$

$= \frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}} \times \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$

$= \frac{\left(1+\sqrt{3}\right)^2}{\left(\sqrt{3}-1\right)^2} = \left(\frac{\left(1+\sqrt{3}\right)^2}{\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)}\right)^2 = \frac{\left(1+\sqrt{3}\right)^4}{4}$

3 $\sin\theta = \frac{24}{25} \quad 0 < \theta < \frac{\pi}{2}$

$\cos^2\theta = 1 - \left(\frac{24}{25}\right)^2 = \frac{49}{625} \quad \therefore \cos\theta = \frac{7}{25} \quad \tan\theta = \frac{24}{7}$

$\sin\phi = \frac{3}{5} \quad \frac{\pi}{2} < \phi < \pi \Rightarrow \cos\phi = \frac{-4}{5} \quad \tan\phi = \frac{-3}{4}$

$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} = \frac{\frac{24}{7}-\frac{3}{4}}{1-\frac{24}{7}\times\frac{-3}{4}}$

$= \frac{\frac{75}{28}}{\frac{100}{28}} = \frac{75}{100} = \frac{3}{4}$

4 $\cot(A+B) = \frac{1}{\tan(A+B)} = \frac{1-\tan A\tan B}{\tan A + \tan B} \times \frac{\frac{1}{\tan A\tan B}}{\frac{1}{\tan A\tan B}}$

$= \frac{\frac{1}{\tan A\tan B}-1}{\frac{1}{\tan B}+\frac{1}{\tan A}} = \frac{\cot A\cot B - 1}{\cot A + \cot B} \quad \text{(QED)}$

5 **a** $\frac{\sin(A+B)}{\cos A\cos B} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B}$

$= \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} = \tan A + \tan B \quad \text{(QED)}$

b $\left(\sin A + \cos A\right)\left(\sin B + \cos B\right)$

$\equiv \sin A\sin B + \sin A\cos B + \cos A\sin B$

$+ \cos A\cos B$

$\equiv \left(\sin A\cos B + \cos A\sin B\right)$

$+ \left(\sin A\sin B + \cos A\cos B\right)$

$\equiv \sin\left(A+B\right) + \cos\left(A-B\right) \quad \text{(QED)}$

6 **a** Let $\alpha = \arctan\left(\frac{1}{4}\right)$ and $\beta = \arctan\left(\frac{3}{5}\right)$

Then $\tan\alpha = \frac{1}{4}$ and $\tan\beta = \frac{3}{5}$

$$\therefore \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$= \frac{\frac{1}{4}+\frac{3}{5}}{1-\frac{1}{4}\times\frac{3}{5}} = \frac{\frac{17}{20}}{\frac{17}{20}} = 1$$

$\therefore \alpha + \beta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

But $0 < \alpha < \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so $0 < \alpha + \beta < \frac{3\pi}{4}$

$\therefore \alpha + \beta = \frac{\pi}{4}$ i.e. $\arctan\left(\frac{1}{4}\right) + \arctan\left(\frac{3}{5}\right) = \frac{\pi}{4}$ (QED).

b Let arctan (4) = γ

Then $\tan\gamma = 4 = \frac{1}{\tan\alpha}$

$\therefore \gamma = \frac{\pi}{2} - 2$

Similarly if $\delta = \arctan\left(\frac{5}{3}\right)$, then $\delta = \frac{\pi}{2} - \beta$

$$\therefore \arctan(4) + \arctan\left(\frac{5}{3}\right) = \frac{\pi}{2} - \alpha + \frac{\pi}{2} - \beta$$

$$= \pi - (\alpha+\beta)$$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

Exercise 8E

1 **a** $\sin(A + B) = \sin A\cos B + \cos A\sin B$

Let $A = B = \theta$

$\sin 2\theta = \sin\theta\cos\theta + \cos\theta\sin\theta$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta$ (QED)

b $\cos(A + B) = \cos A\cos B - \sin A\sin B$

Let $A = B = \theta$

$\cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ (QED)

c $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

Let $A = B = \theta$, $\tan 2\theta = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$

$\therefore \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ (QED)

2 $\cos\alpha = \frac{4}{5}$ $\cos\beta = \frac{7}{25}$

$\sin\alpha = \pm\frac{3}{5}$ $\sin\beta = \pm\frac{24}{25}$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{4}{5}\times\frac{7}{25} \pm \frac{3}{5}\times\frac{24}{25}$$

$$= \frac{100}{125} \text{ or } \frac{-44}{125} = \frac{4}{5} \text{ or } \frac{-44}{125}$$

3 $\cos A = \frac{1}{3}$, $\cos 2A = 2\cos^2 A - 1 = 2\left(\frac{1}{3}\right)^2 - 1 = \frac{-7}{9}$

$\cos 4A = 2\cos^2 2A - 1 = 2\left(\frac{-7}{9}\right)^2 - 1 = \frac{17}{81}$

4 $$\tan\left(\theta + \frac{\pi}{3}\right)\tan\left(\theta - \frac{\pi}{3}\right) = \frac{\left(\tan\theta + \sqrt{3}\right)}{\left(1 - \sqrt{3}\tan\theta\right)}\frac{\left(\tan\theta - \sqrt{3}\right)}{\left(1 + \sqrt{3}\tan\theta\right)}$$

$$= \frac{\tan^2\theta - 3}{1 - 3\tan^2\theta} \quad \text{(QED)}$$

5 **a** $2\cos^2 A + \sin^2 A = \cos^2 A + 1 = \frac{1}{2}(\cos 2A + 1) + 1$

$$= \frac{1}{2}(\cos 2A + 3)$$

b $\cos^4 A = (\cos^2 A)^2 = \frac{1}{4}(\cos 2A + 1)^2$

c $\sin^4 A = (\sin^2 A)^2 = \frac{1}{4}(1 - \cos 2A)^2$

6 **a** $(1 + \tan^2\theta)(1 - \cos 2\theta) = (1 + \tan^2\theta)(1 - (1 - 2\sin^2\theta))$

$$= (1 + \tan^2\theta)\ 2\sin^2\theta$$

$$= \sec^2\theta\ 2\sin^2\theta$$

$$= \frac{2\sin^2\theta}{\cos^2\theta}$$

$$= 2\tan^2\theta \quad \text{(QED)}$$

b $\left(1 + \tan^2\theta\right)\left(1 + \cos 2\theta\right) = \sec^2\theta\left(1 + 2\cos^2\theta - 1\right)$

$$= \frac{1}{\cos^2\theta}(2\cos^2\theta)$$

$$= 2 \quad \text{(QED)}$$

7 **a** $\frac{1-\cos 2A}{1+\cos 2A} = \frac{1-(1-2\sin^2 A)}{1+(2\cos^2 A - 1)} = \frac{2\sin^2 A}{2\cos^2 A} = \tan^2 A$ (QED)

b $$\frac{1-\tan^2 A}{1+\tan^2 A} = \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}} = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

$$= \frac{\cos 2A}{1} = \cos 2A \quad \text{(QED)}$$

c $$\frac{\sin 2A}{1-\cos 2A} = \frac{2\sin A\cos A}{1-\left(1-2\sin^2 A\right)} = \frac{2\sin A\cos A}{2\sin^2 A}$$

$$= \frac{\cos A}{\sin A} = \cot A \quad \text{(QED)}$$

d $\cos 3A = \cos(A + 2A)$

$$= \cos A\cos 2A - \sin A\sin 2A$$

$$= \cos A\left(2\cos^2 A - 1\right) - \sin A(2\sin A\cos A)$$

$$= 2\cos^3 A - \cos A - 2\sin^2 A\cos A$$

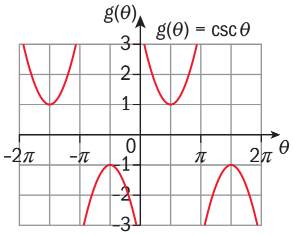
$$= 2\cos^3 A - \cos A - 2\cos A\left(1 - \cos^2 A\right)$$

$$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$$

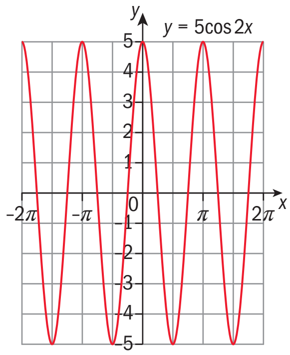
$$= 4\cos^3 A - 3\cos A \quad \text{(QED)}$$

Exercise 8F

1 **a**



2 **a**



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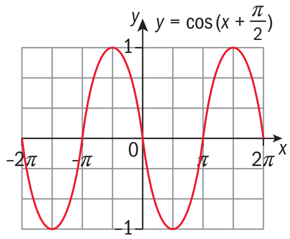

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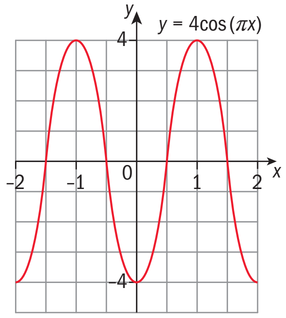

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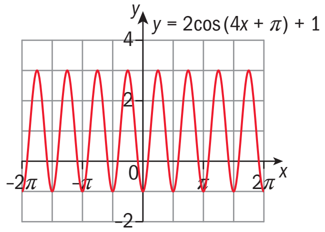
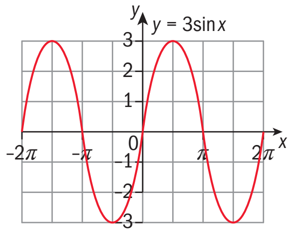


3 **a**



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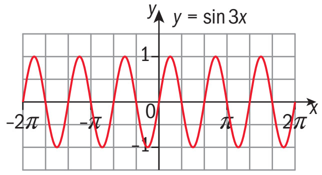

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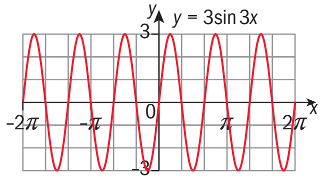

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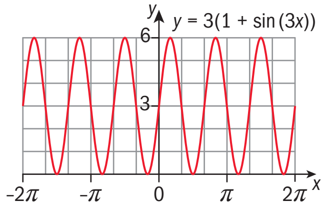
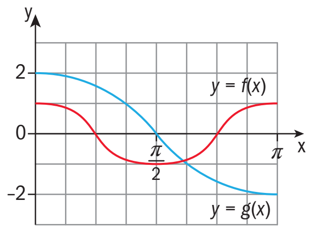


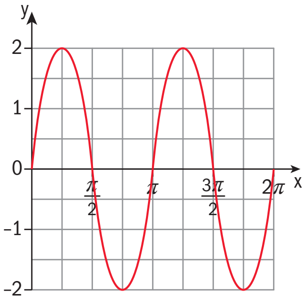
4



$f(x) = g(x)$, 1 solution

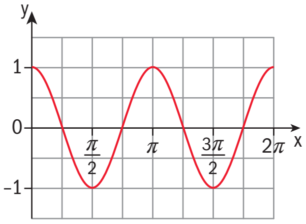
5 **a** $f(x) = 4\sin x \cos x = 2\sin 2x$

$f(x)$ is odd, period $= \pi$



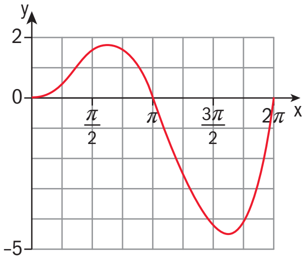
b $g(x) = 1 - 2\sin^2 x = \cos^2 x$

$g(x)$ is even, period $= \pi$



c $h(x) = x \sin x$

$h(x)$ is even, not periodic



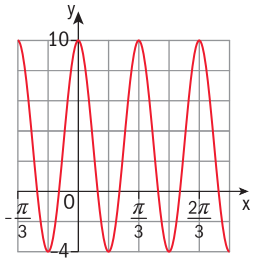
Exercise 8G

1 **i** $f(x) = 7\sin\left[6\left(x - \frac{\pi}{12}\right)\right] + 3$

a amplitude $= 7$ period $= \frac{2\pi}{6} = \frac{\pi}{3}$

phase shift $= \frac{\pi}{12}$

b min. value $= -4$, max. value $= 10$

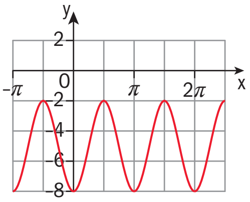


ii $f(x) = -3\sin\left(2x + \frac{\pi}{2}\right) - 5 = -3\sin\left[2\left(x + \frac{\pi}{4}\right)\right] - 5$

a amplitude $= 3$ period $= \frac{2\pi}{2} = \pi$

phase shift $= \frac{\pi}{4}$

b min value $= -8$ max value $= -2$

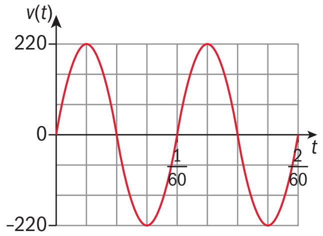


2 $V(t) = 220\sin(120\pi t)$

a max $= 220$ **b** min $= -220$

c amplitude $= 220$ **d** period $= \frac{2\pi}{120\pi} = \frac{1}{60}$

e $V(t)$



3 $h(t) = a\sin\left[b\left(t + c\right)\right] + d$

a $a = \frac{14.4 - 1.2}{2} = 6.6$

$d = \frac{14.4 + 1.2}{2} = 7.8$

$\frac{2\pi}{b} = 12 \quad \therefore b = \frac{\pi}{6}$

$h(t) = 6.6\sin\left[\frac{\pi}{6}(t + c)\right] + 7.8$

$h(8.25) = 14.4$

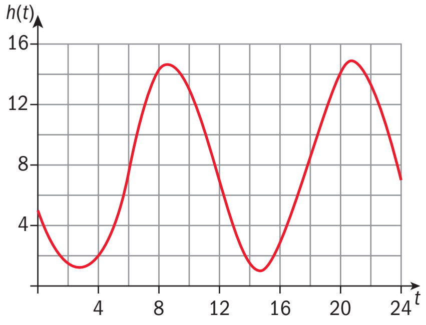
$\therefore 14.4 = 6.6\sin\left[\frac{\pi}{6}(8.25 + c)\right] + 7.8$

$\therefore \sin\left[\frac{\pi}{6}(8.25 + c)\right] = 1 \quad \therefore \frac{\pi}{6}(8.25 + c) = \frac{\pi}{2}$

$\therefore 8.25 + c = 3 \quad \therefore c = -5.25$

$a = 6.6 \quad b = \frac{\pi}{6} \quad c = -5.25 \quad d = 7.8$

b



$h(t) = 6.6\sin\left[\frac{\pi}{6}(t - 5.25)\right] + 7.8$

$1.2 = 6.6\sin\left[\frac{\pi}{6}(t - 5.25)\right] + 7.8$

$\sin\left[\frac{\pi}{6}(t - 5.25)\right] = -1$

$\frac{\pi}{6}(t - 5.25) = \frac{-\pi}{2}$

$t - 5.25 = -3$

$\therefore t = 2.25$

First low tide is at 02:15

c Points of intersection: $(0.086757, 5), (4.413243, 5),$
$(12.086757, 5), (16.413243, 5)$

$h(t) \geq 5$ for $0 \leq t \leq 0.086757,$

$4.413243 \leq t \leq 12.086757$, and $16.413243 \leq t \leq 2^4$

Time intervals are: 00 : 00 to 00 : 05, 04 : 25 to 12 : 05
and 16 : 25 to 24 : 00

4 $f(x) = a\sin[b(x + c)] + d$

$a = \frac{12.75 - 10.65}{2} = 1.05$

$d = \frac{12.75 + 10.65}{2} = 11.7$

$\frac{2\pi}{b} = 365 \quad \therefore b = \frac{2\pi}{365}$

$f(x) = 1.05\sin\left[\frac{2\pi}{365}(x + c)\right] + 11.7$

On 21 June, $x = 172$, $f(172) = 12.75$

$12.75 = 1.05\sin\left[\frac{2\pi}{365}(172 + c)\right] + 11.7$

$\sin\left[\frac{2\pi}{365}(172 + c)\right] = 1 \quad \therefore \frac{2\pi}{365}(172 + c) = \frac{\pi}{2}$

$\therefore 172 + c = \frac{365}{4} \quad \therefore c = -80.75$

$a = 1.05 \quad b = \frac{2\pi}{365} \quad c = -80.75 \quad d = 11.7$

On 4 July, $x = 185$, $f(185) = 12.72$

$\therefore 12.7$ hours

Exercise 8H

1 **a** $\cos\left(\arcsin\frac{\sqrt{2}}{2}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

b $\sec\left(\arctan\frac{1}{2}\right)$. Let $\theta = \arctan\frac{1}{2} \therefore \tan\theta = \frac{1}{2}$

$\sec^2\theta = 1 + \tan^2\theta = \frac{5}{4} \quad \therefore \sec\theta = \frac{\sqrt{5}}{2}$

$\therefore \sec\left(\arctan\frac{1}{2}\right) = \frac{\sqrt{5}}{2}$

c $\cos\left(\arcsin\left(-\frac{\sqrt{3}}{2}\right)\right) = \cos\left(\frac{-\pi}{3}\right) = \frac{1}{2}$

d $\tan\left(\arctan\frac{5\pi}{6}\right) = \frac{5\pi}{6}$

e $\arccos\left(\sin\left(\frac{3\pi}{4}\right)\right) = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

f $\arcsin\left(\sin\left(\frac{-7\pi}{6}\right)\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

2 **a** $\sin\left(\arcsin\frac{1}{2}+\arccos\frac{1}{2}\right)=\sin\left(\frac{\pi}{6}+\frac{\pi}{3}\right)=\sin\frac{\pi}{2}=1$

b Let $\arcsin\frac{3}{5}=\theta \quad \therefore \sin\theta=\frac{3}{5}, \cos\theta=\frac{4}{5}$

$\arccos\left(\frac{-4}{5}\right)=\phi \quad \therefore \cos\phi=\frac{-4}{5}, \sin\phi=\frac{3}{5}$

$\cos\left(\arcsin\frac{3}{5}-\arccos\left(\frac{-4}{5}\right)\right)=\cos(\theta-\phi)$

$=\cos\theta\cos\phi+\sin\theta\sin\phi$

$=\frac{4}{5}\times\frac{-4}{5}+\frac{3}{5}\times\frac{3}{5}=\frac{-7}{25}$

c Let $\arctan\frac{3}{4}=\theta, \quad \tan\theta=\frac{3}{4}$

$\tan\left(2\arctan\left(\frac{3}{4}\right)\right)=\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$

$=\frac{\frac{3}{2}}{1-\frac{9}{16}}=\frac{3}{2}\times\frac{16}{7}=\frac{24}{7}$

3 **a** Let $\arcsin a=\theta \quad \therefore \sin\theta=a \quad \cos\theta=\sqrt{1-a^2}$

$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{a}{\sqrt{1-a^2}}$

$\therefore \tan(\arcsin a)=\frac{a}{\sqrt{1-a^2}}$ (QED)

b Let $\arcsin a=\theta$ and $\arccos a=\phi$

$\sin\theta=a \qquad \cos\phi=a$

$\cos(\arcsin a+\arccos a)=\cos(\theta+\phi)$

$=\cos\theta\cos\phi-\sin\theta\sin\phi$

$=\sqrt{1-a^2}(a)-a\sqrt{1-a^2}$

$=0$ (QED)

c Let $\arccos a=\theta \therefore \cos\theta=a, \sin\theta=\sqrt{1-a^2}$

$\tan(\arccos a)=\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{\sqrt{1-a^2}}{a}$ (QED)

Exercise 8I

1 $3\sin x=2\tan x \qquad -\pi\le x\le\pi$

$3\sin x=\frac{2\sin x}{\cos x}$

$3\sin x\cos x-2\sin x=0$

$\sin x(3\cos x-2)=0$

$\sin x=0$ or $\cos x=\frac{2}{3}$

$x=0, \pm\pi$ or $x=\pm0.841$

2 $\cot\theta+\sin\theta=6 \qquad [0,\pi]$

$\frac{\cos\theta}{\sin\theta}+\sin\theta=6$

Using GDC: $\theta=0.170$

3 $3\cos2\theta=2\cos^2\theta \quad [-\pi,\pi]$

$3(2\cos^2\theta-1)=2\cos^2\theta$

$4\cos^2\theta=3 \therefore \cos^2\theta=\frac{3}{4}, \cos\theta=\pm\frac{\sqrt{3}}{2}$

$\theta=\pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$

4 $3\tan^2\theta-\frac{14}{\cos\theta}+18=0$

$3(\sec^2\theta-1)-14\sec\theta+18=0$

$3\sec^2\theta-14\sec\theta+15=0$

$(3\sec\theta-5)(\sec\theta-3)=0$

$\sec\theta=\frac{5}{3}$ or 3

5 $\sin x-\cos x=1 \qquad 0\le x\le\pi$

$2\sin\frac{x}{2}\cos\frac{x}{2}-\left(2\cos^2\frac{x}{2}-1\right)=1$

$2\sin\frac{x}{2}\cos\frac{x}{2}-2\cos^2\frac{x}{2}=0$

$2\cos\frac{x}{2}\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)=0$

$\cos\frac{x}{2}=0$ or $\sin\frac{x}{2}=\cos\frac{x}{2}\Rightarrow\tan\frac{x}{2}=1$

$\frac{x}{2}=\frac{\pi}{2}$ or $\frac{x}{2}=\frac{\pi}{4}$

$\therefore x=\frac{\pi}{2}$ or π

6 $\cos\theta+\sin\theta=2 \qquad -\pi\le\theta\le\pi$

$\frac{1}{\sin\theta}+\sin\theta=2$

$1+\sin^2\theta=2\sin\theta$

$\sin^2\theta-2\sin\theta+1=0$

$(\sin\theta-1)^2=0$

$\sin\theta=1 \therefore \theta=\frac{\pi}{2}$

7 $\frac{\sin x-3\cos x}{\sin x-\cos x}=7$

$\sin x-3\cos x=7\sin x-7\cos x$

$4\cos x=6\sin x$

$\therefore \tan x=\frac{4}{6}=\frac{2}{3}$

a $\tan2x=\frac{2\tan x}{1-\tan^2 x}$

$=\frac{\frac{4}{3}}{1-\frac{4}{9}}=\frac{4}{3}\times\frac{9}{5}=\frac{12}{5}$

b $\tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$

$\frac{2}{3}=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$

$2-2\tan^2\frac{x}{2}=6\tan\frac{x}{2}$

$\tan^2\frac{x}{3}+3\tan\frac{x}{2}-1=0$

$\tan\frac{x}{2}=\frac{-3\pm\sqrt{9-(-4)}}{2}=\frac{-3\pm\sqrt{13}}{2}$

8 $\frac{x}{2}\sin2x=\sqrt{x}\sin x \qquad 0\le x\le2\pi$

From graph, $x=0, \pi, 5.17, 2\pi$

9 $-5x^2\cos8x=\tan x \qquad 0\le x\le\frac{\pi}{2}$

Using GDC, $x=0, 0.294, 0.536, 1.02, 1.32$

Exercise 8J

1 **a** $QR^2 = 5^2 + 8^2 - 2\times 5\times 8\cos 30° = 19.17...$

$QR = 4.44$

$\cos Q = \frac{5^2+4.44^2-8^2}{2\times 5\times 4.44} = -0.4342\ ...$

$P\hat{Q}R = 11.6° \quad P\hat{R}Q = 34.3°$

b $XZ^2 + 4^2 + 5^2 - 2\times 4\times 5\cos 95° = 44.48...$

$XZ = 6.67$

$\cos Z = \frac{4^2+6.67^2-5^2}{2\times 4\times 6.67} = 0.6650...$

$X\hat{Z}Y = 48.3° \quad Y\hat{X}Z = 36.7°$

c $\cos A = \frac{4^2+8^2-5^2}{2\times 4\times 8} = 0.859375$

$B\hat{A}C = 30.8°$

$\cos C = \frac{8^2+5^2-4^2}{2\times 8\times 5} = 0.9125$

$A\hat{C}B = 24.1° \quad A\hat{B}C = 125°$

2 $\cos A = \frac{3.9^2+2.3^2-4.5^2}{2\times 3.9\times 2.3} = 0.01393...$

$A = 89.2°$, largest angle $= 89.2°$

3 $\cos P = \frac{3^2+4^2-2^2}{2\times 3\times 4} = 0.875$

$P = 29.0°$, smallest angle $= 29.0°$

4 $(2x-1)^2 = x^2 + 5^2 - 2x^5\cos 60°$

$4x^2 - 4x + 1 = x^2 + 25 - 5x$

$3x^2 + x - 24 = 0$

$(3x-8)(x+3) = 0$

$x = \frac{8}{3}$

$2x - 1 = \frac{13}{3},\ \cos B = \frac{5^2+\left(\frac{13}{3}\right)^2-\left(\frac{8}{3}\right)^2}{2\times 5\times \frac{13}{3}} = 0.8461...$

$A\hat{B}C = 32.2° \quad A\hat{C}B = 87.8°$

5 In ΔABC, $p^2 = a^2 + b^2 - 2ab\cos A\hat{B}C$

In ΔABD, $q^2 = a^2 + b^2 - 2ab\cos B\hat{A}D$

$B\hat{A}D = 180° - A\hat{B}C$

$\therefore \cos B\hat{A}D = \cos\left(180° - A\hat{B}C\right) = -\cos A\hat{B}C$

$\therefore q^2 = a^2 + b^2 + 2ab\cos A\hat{B}C$

$\therefore p^2 + q^2 = 2\left(a^2 + b^2\right)$ (QED)

Exercise 8K

1 **a** $A\hat{C}B = 180° - (30° + 125°) = 25°$

$A\hat{C}B = 25°$

$\frac{AC}{\sin 125°} = \frac{10}{\sin 30°} \quad \therefore AC = 16.4\text{ cm}$

$\frac{AB}{\sin 25°} = \frac{10}{\sin 30°} \quad \therefore AB = 8.45\text{ cm}$

b $P\hat{Q}R = 95°$

$\frac{RP}{\sin 95°} = \frac{7}{\sin 45°} \quad \therefore RP = 9.86\text{ cm}$

$\frac{QR}{\sin 95°} = \frac{7}{\sin 45°} \quad \therefore QR = 6.36\text{ cm}$

c $\frac{\sin A}{7} = \frac{\sin 40°}{9} \quad \therefore \sin A = 0.4999...$

A = 29.996° or 150.004° (only the acute angle is possible as this angle is opposite side 7 and therefore smaller than 40°)

$B\hat{A}C = 30.0° \quad A\hat{B}C = 110°$

$\frac{AC}{\sin 110°} = \frac{9}{\sin 40°} \quad \therefore AC = 13.2\text{ cm}$

2 $\frac{\sin Q}{80} = \frac{\sin 15°}{150} \quad \therefore \sin Q = 0.1380$

$Q = 7.934°$

$R = 180° - (15° + 7.93°) = 157.07°$

$\frac{PQ}{\sin 157.07°} = \frac{150}{\sin 15°} \quad \therefore PQ = 225.84\text{ km}$

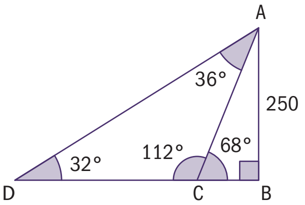
extra distance travelled = 230 − 225.84

= 4.16... km

time lost $= \frac{4.16..}{400}$ hours $= 0.0104..$ hours

= 37 sec (nearest second)

3



In ΔABC,

$\sin 68° = \frac{250}{AC} \quad \therefore AC = 269.63...$

$\frac{CD}{\sin 36°} = \frac{269.63..}{\sin 32°}$

$\therefore CD = 299 \quad \therefore$ length of lake = 299 m (nearest m)

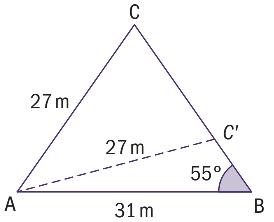
4 In Δ MBC, $M\hat{B}C = 116°$, $B\hat{M}C = 41°$

$\frac{MC}{\sin 116°} = \frac{15}{\sin 41°} \quad \therefore MC = 20.5\text{ m}$

$\frac{MC}{\sin 23°} = \frac{15}{\sin 41°} \quad \therefore MB = 8.93\text{ m}$

In Δ ABM, $\sin 64° = \frac{MA}{8.93...} \quad \therefore MA = 8.03\text{ m}$

5



$\frac{\sin C}{31} = \frac{\sin 55}{27} \quad \therefore \sin C = 0.9405$

$\therefore C = 70.1°$ or $109.9°$

$A = 55°, C = 70°$ or $A = 15°, C = 110°$

Exercise 8L

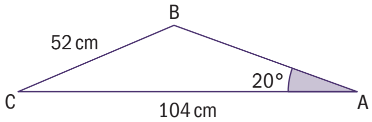
1 $PR^2 = 10^2 + 13^2 - 2\times 10\times 13\cos 125° = 418.129...$

$PR = 20.448..$

Area $= \frac{1}{2}\times 10\times 13\sin 125° + \frac{1}{2}\times 15\times 20.448\sin 70°$

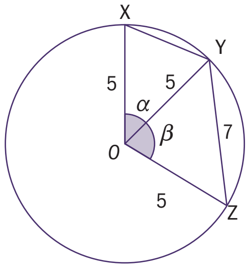
$= 197$ sq.units

2



$$\frac{\sin \text{B}}{104} = \frac{\sin 20°}{52}$$
$$\sin \text{B} = 0.68404...$$
$$\text{B} = 43.16.. \text{ or } 136.83...$$
$$\text{C} = 116.83.. \text{ or } 23.16...$$
$$\text{Area}_1 = \frac{1}{2} \times 52 \times 104 \sin 116.83.. = 2412.7...$$
$$\text{Area}_2 = \frac{1}{2} \times 52 \times 104 \sin 23.16.. = 1063.49...$$
$$\therefore \text{difference} = 1349 \text{ or } 1350\,\text{cm}^2 (3\,\text{sf})$$

3



$$\text{In } \Delta\text{OXY}, \cos\alpha = \frac{5^2+5^2-3^2}{2\times5\times5} = 0.82$$
$$\alpha = 34.915°...$$
$$\text{In } \Delta\text{OYZ}, \cos\beta = \frac{5^2+5^2-7^2}{2\times5\times5} = 0.02$$
$$\beta = 88.854°...$$
$$\text{Area OXYZ} = \frac{1}{2}\times5\times5\sin34.915° + \frac{1}{2}\times5\times5\sin88.854°$$
$$= 19.7\,\text{cm}^2$$

4 $\text{In } \Delta\text{ABC}, \tan 60° = \frac{12}{AC} \quad \therefore \text{AC} = 6.928$

$\text{In } \Delta\text{ABD}, \tan 55° = \frac{12}{AD} \quad \therefore \text{AD} = 8.402$

$\text{In } \Delta\text{ACD}, \cos\text{C}\hat{\text{A}}\text{D} = \frac{6.928^2 + 8.402^2 - 15^2}{2\times6.928\times8.402} = -0.9138...$

$\therefore \text{C}\hat{\text{A}}\text{D} = 156°$

$\text{Area } \Delta\text{CAD} = \frac{1}{2}\times6.928\times8.402\times\sin156° = 11.8\,\text{m}^2$

5 **a** $\text{area } \Delta\text{POQ} = \frac{1}{2}r^2\sin\frac{2\pi}{3} = \frac{1}{2}r^2\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}r^2$

$\text{area } \Delta\text{ROS} = \frac{1}{2}r^2\sin\frac{\pi}{6} = \frac{1}{4}r^2$

b $\text{area} = \frac{1}{2}r^2\frac{2\pi}{3} - \frac{\sqrt{13}}{4}r^2 = r^2\left(\frac{\pi}{3} - \frac{\sqrt{13}}{4}\right) = \frac{r^2}{12}\left(4\pi - 3\sqrt{13}\right)$

c $\text{area} = \frac{1}{2}r^2\frac{\pi}{6} - \frac{1}{4}r^2 = r^2\left(\frac{\pi}{12} - \frac{1}{4}\right) = \frac{r^2}{12}(\pi - 3)$

d $\text{shaded area} = \frac{r^2}{12}\left(4\pi - 3\sqrt{3}\right) - \frac{r^2}{12}(\pi - 3)$

$= \frac{r^2}{12}\left(3\pi - 3\sqrt{3} + 3\right) = \frac{r^2}{4}\left(\pi + 1 - \sqrt{3}\right) \quad \text{(QED)}$

Review exercise

1 $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2} \quad \text{(QED)}$

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 2\left(\frac{1}{\sqrt{1+t^2}}\right) - 1 = \frac{2}{1+t^2} - 1$

$= \frac{2-(1+t^2)}{1+t^2} = \frac{1-t^2}{1+t^2} \quad \text{(QED)}$

$$\sqrt{3}\sin\theta + \cos\theta = 1 \quad 0 \le \theta \le 2\pi$$
$$\frac{2\sqrt{3}t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1$$
$$2\sqrt{3}t + 1 - t^2 = 1 + t^2$$
$$2t^2 - 2\sqrt{3}t = 0$$
$$2t\left(t - \sqrt{3}\right) = 0$$
$$\tan\frac{\theta}{2} = 0 \text{ or } \tan\frac{\theta}{2} = \sqrt{3}$$
$$\frac{\theta}{2} = 0 \text{ or } \pi \text{ or } \frac{\theta}{2} = \frac{\pi}{3}$$
$$\theta = 0, \frac{2\pi}{3} \text{ or } 2\pi$$

2 **a** $\sin165° = \sin(120° + 45°)$

$= \sin120°\cos45° + \cos120°\sin45°$

$= \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} - \frac{1}{2}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$

b $\tan105° = \tan(60° + 45°)$

$= \frac{\tan60° + \tan45°}{1 - \tan60°\tan45°}$

$= \frac{\sqrt{3}+1}{1-\sqrt{3}}$

c $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$

$= \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} - \frac{1}{2}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$

d $\tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

$1 - \tan^2\frac{\pi}{8} = 2\tan\frac{\pi}{8}$

$\tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$

$\tan\frac{\pi}{8} = \frac{-2\pm\sqrt{4+4}}{2} = \frac{-2\pm\sqrt{8}}{2} = -1 + \sqrt{2} \quad \text{since } \tan\frac{\pi}{8} > 0$

$\therefore \tan\frac{\pi}{8} = \sqrt{2} - 1$

3 **a** $\frac{1}{1-\tan\theta} = \frac{1}{1-\frac{\sin\theta}{\cos\theta}} = \frac{\cos\theta}{\cos\theta - \sin\theta}$

$\therefore \frac{\cos\theta}{\cos\theta - \sin\theta} = \frac{1}{1-\tan\theta} \quad \text{(QED)}$

b $\frac{\cos(A-B)}{\cos A\cos B} = \frac{\cos A\cos B + \sin A\sin B}{\cos A\cos B}$

$= \frac{\cos A\cos B}{\cos A\cos B} + \frac{\sin A\sin B}{\cos A\cos B}$

$= 1 + \tan A\tan B \quad \text{(QED)}$

c $\cos3A = \cos(A + 2A) = \cos A\cos2A - \sin A\sin2A$

$= \cos A\left(2\cos^2 A - 1\right) - \sin A2\sin A\cos A$

$= 2\cos^2 A - \cos A - 2\sin^2\cos A$

$\sin3A = \sin(A + 2A) = \sin A\cos2A + \cos A\sin2A$

$= \sin A\left(2\cos^2 A - 1\right) + \cos A2\sin A\cos A$

$= 2\sin A\cos^2 A - \sin A + 2\sin A\cos^2 A$

$= 4\sin A\cos^2 A - \sin A$

$$\cos 3A - \sin 3A = 2\cos^3 A - \cos A - 2\sin^2 A\cos A$$
$$- 4\sin A\cos^2 A + \sin A$$
$$= 2\cos A\left(1 - \sin^2 A\right) - \cos A - 2\sin^2 A\cos A$$
$$- 4\sin A\cos^2 A + \sin A$$
$$= \cos A - 4\sin^2 A\cos A - 4\sin A\cos^2 A + \sin A$$
$$= \cos A(1 - 4\sin A\cos A) + \sin A(1 - 4\sin A\cos A)$$
$$= (\cos A + \sin A)(1 - 4\sin A\cos A) \quad \text{(QED)}$$

d $2\sin 2\theta\left(1-2\sin^2\theta\right) = 2\sin 2\theta\cos 2\theta$
$$= \sin 4\theta \quad \text{(QED)}$$

e $1+2\cos 2A+\cos 4A = 1+2\cos 2A+2\cos^2 2A-1$
$$= 2\cos 2A\left(1+\cos 2A\right)$$
$$= 2\cos 2A\left(1+2\cos^2 A-1\right)$$
$$= 4\cos^2 A\cos 2A \quad \text{(QED)}$$

4 a Let $\arcsin\frac{3}{5} = \theta, \sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}$

$\arccos\frac{1}{2} = \phi, \cos\phi = \frac{1}{2}, \sin\phi = \frac{\sqrt{3}}{2}$

$\cos\left(\arcsin\frac{3}{5} - \arccos\frac{1}{2}\right) = \cos(\theta-\phi)$

$= \cos\theta\cos\phi + \sin\theta\sin\phi$

$= \frac{4}{5}\times\frac{1}{2}+\frac{3}{5}\times\frac{\sqrt{3}}{2} = \frac{4+3\sqrt{3}}{10}$

b Let $\arccos\left(\frac{-3}{5}\right) = \theta \quad \therefore \cos\theta = \frac{-3}{5}, \sin\theta = \frac{4}{5}$

$\sin\left[2\arccos\left(\frac{-3}{5}\right)\right] = \sin(2\theta) = 2\sin\theta\cos\theta$

$= 2\left(\frac{4}{5}\right)\left(\frac{-3}{5}\right) = \frac{-24}{25}$

c $\arctan(-1) = \frac{-\pi}{4}$

let $\arccos\left(\frac{-4}{5}\right) = \theta \quad \therefore \cos\theta = \frac{-4}{5}, \sin\theta = \frac{3}{5}$

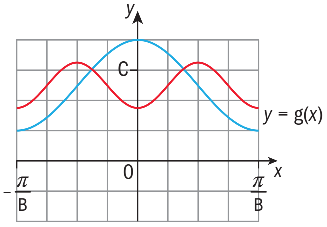
$\sin\left[\arctan(-1) + \arccos\left(\frac{-4}{5}\right)\right] = \sin\left(\theta - \frac{\pi}{4}\right)$

$= \sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}$

$= \frac{3}{5}\times\frac{\sqrt{2}}{2}+\frac{4}{5}\times\frac{\sqrt{2}}{2}$

$= \frac{7\sqrt{2}}{10}$

5



6 Let $\arcsin x = \theta, \sin\theta = x, \cos\theta = \sqrt{1-x^2}$

$\arccos x = \phi, \cos\phi = x, \sin\phi = \sqrt{1-x^2}$

$\sin[\arcsin x - \arccos x] = \sin(\theta-\phi)$

$= \sin\theta\cos\phi - \cos\theta\sin\phi$

$= x^2 - \left(\sqrt{1-x^2}\right)^2 = x^2-(1-x^2) = 2x^2-1 \quad \text{(QED)}$

$\arcsin x - \arccos x = \arcsin(1-x)$

$\therefore \sin\left(\arcsin(1-x)\right) = 2x^2-1$

$1-x = 2x^2-1$

$2x^2+x-2 = 0$

$x = \frac{-1\pm\sqrt{1+16}}{4}$

$x = \frac{-1+\sqrt{17}}{4}$

$\arcsin x$ acute $\Rightarrow x \geq 0$
$\arcsin(1-x)$ acute $\Rightarrow 1-x \geq 0$
$\Rightarrow x \leq 1$
$\therefore 0 \leq x \leq 1$

$x = \frac{1}{4}\left(\sqrt{17}-1\right) \quad \text{(QED)}$

7 $\tan(2x+y) = \frac{\tan 2x + \tan y}{1-\tan 2x\tan y}$

$\tan\left(\frac{\pi}{4}\right) = \frac{\tan 2x + \tan y}{1-\tan 2x\tan y} = 1$

$\therefore \tan 2x + \tan y = 1 - \tan 2x\tan y$

$\tan y(1+\tan 2x) = 1 - \tan 2x$

$\tan y = \frac{1-\tan 2x}{1+\tan 2x}$

$\tan y = \frac{1-\frac{2\tan x}{1-\tan^2 x}}{1+\frac{2\tan x}{1-\tan^2 x}} = \frac{1-\tan^2 x - 2\tan x}{1-\tan^2 x+2\tan x}$

$\therefore \tan y = \frac{1-2\tan x-\tan^2 x}{1+2\tan x-\tan^2 x} \quad \text{(QED)}$

Review exercise

1 $\cos(A-B) - \cos(A+B) = \cos A\cos B + \sin A\sin B$
$$-(\cos A\cos B - \sin A\sin B)$$
$$= 2\sin A\sin B \quad \text{(QED)}$$

$\sin 3x\sin x = -1$

Let $A = 3x, B = x$

$\cos 2x - \cos 4x = 2\sin 3x\sin x$

$\cos 2x - \cos 4x = -2$

$\cos 2x - \left(2\cos^2 2x - 1\right) = -2$

$\cos 2x - 2\cos^2 2x + 1 = -2$

$2\cos^2 2x - \cos 2x - 3 = 0$

$(2\cos 2x - 3)(\cos 2x + 1) = 0$

$\cos 2x = -1$

$2x = \pi$

$x = \frac{\pi}{2}$

2 a $\sin y + \sin x = 1.1 \Rightarrow y = \arcsin(1.1 - \sin x)$

$\cos y + \sin 2x = 1.8 \Rightarrow y = \arccos(1.8 - \sin 2x)$

b Using GDC, $x = 0.619,\ y = 0.546$
or $x = 1.09,\ y = 0.216$

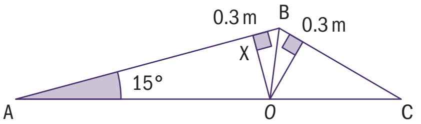
3 **a** $\mathrm{A\hat{D}B} = 110° \quad \therefore \mathrm{A\hat{B}O} = 180° - (15° + 110°) = 55°$

$\mathrm{O\hat{B}C} = \frac{1}{2}(180° - 70°) = 55°$

$\therefore \mathrm{A\hat{B}C} = 55° + 55° = 110°$

b In $\Delta\,\mathrm{ABC}, \frac{\mathrm{AB}}{\sin 55°} = \frac{0.6}{\sin 15°} \quad \therefore \mathrm{AB} = 1.90\,\mathrm{m}$

c



$\mathrm{AX} = 1.898 - 0.3 = 1.598...$

$\tan 15° = \frac{\mathrm{OX}}{1.598...}$

$\therefore \mathrm{OX} = 0.428\,\mathrm{m}$

$\mathrm{radius} = 0.428\,\mathrm{m}$

4 **a** $f(x) = \frac{2 + 3\sin x}{4 + 3\cos x}, \quad 0 \le x \le 2\pi$

For vertical asymptotes,

$4 + 3\cos x = 0$

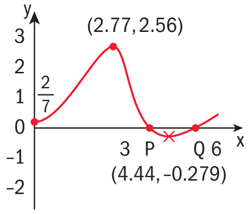
$\cos x = \frac{-4}{3}$ (no solution)

$\therefore$ no vertical asymptotes (QED)

b $f(0) = \frac{2}{7}, \left(0, \frac{2}{7}\right)$

c $p = 3.87 \quad q = 5.55$

d

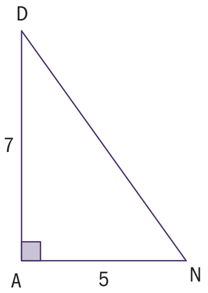


e Points of intersection at $x = 0.510$, 3.53, 3.99, 5.49

$f(x) > g(x)$ for $0.510 < x < 3.53$ and $3.99 < x < 5.49$

f Max. value of $f(x) - g(x)$ is 2.39 (when $x = 1.88$)

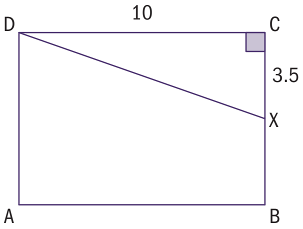
5 **a**



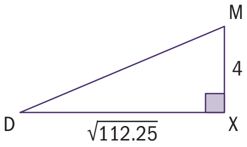
$\mathrm{DN}^2 = 5^2 + 7^2$

$\mathrm{DN} = \sqrt{74} = 8.60$ cm

b



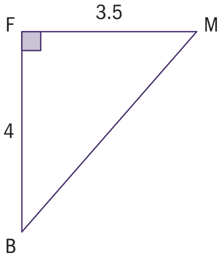
$\mathrm{DX}^2 = 10^2 + 3.5^2 = 112.25$



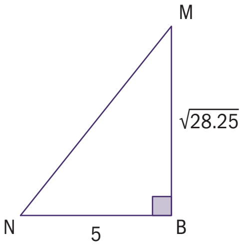
$\mathrm{DM}^2 = 112.25 + 4^2$

$\mathrm{DM} = \sqrt{128.25} = 11.3\,\mathrm{cm}$

c



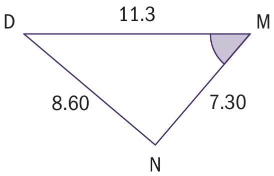
$\mathrm{BM}^2 = 3.5^2 + 4^2 = 28.25$



$\mathrm{NM}^2 = 28.25 + 5^2$

$\mathrm{NM} = \sqrt{53.25} = 7.30\,\mathrm{cm}$

d



$\cos \mathrm{M} = \frac{128.25 + 53.25 - 74}{2\sqrt{128.25}\sqrt{53.25}}$

$= 0.6504$

$\mathrm{D\hat{M}N} = 49.4$

e area $\Delta\,\mathrm{DMN} = \frac{1}{2}\sqrt{128.25}\sqrt{53.25}\sin 49.4°$

$= 31.4\,\mathrm{cm}^2$

6 **a** Area $\Delta\,\mathrm{ABC} = \frac{1}{2}ch$

(h = length of perpendicular from C to AB)

or area $\Delta\,\mathrm{ABC} = \frac{1}{2}ab\sin \mathrm{C}$

$\therefore \frac{1}{2}ch = \frac{1}{2}ab\sin \mathrm{C}$

$\therefore h = \frac{ab}{c}\sin \mathrm{C}$ (QED)

b In ΔBCD, $\tan 30° = \frac{10}{BC}$ $\quad \therefore \text{BC} = 17.3\,\text{m}$

In ΔACD, $\tan 45° = \frac{10}{AC}$ $\quad \therefore \text{AC} = 10\,\text{m}$

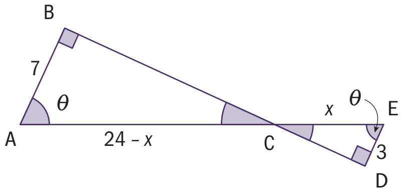
In ΔABC, $\text{AB}^2 = 17.3^2 + 10^2 - 2 \times 17.3$
$\times 10 \cos 150° = 700$

$\text{AB} = 26.5\,\text{m}$

From a, $h = \frac{ab}{c} \sin C = \frac{17.3 \times 10}{26.5} \sin 150°$

$h = 3.27\,\text{m}$

7



$\frac{3}{7} = \frac{x}{24 - x}$

$72 - 3x = 7x$

$x = 7.2\,\text{cm}$

In ΔCDE, $\text{CD}^2 = 7.2^2 - 3^2 = 42.84$,

$\text{CD} = \sqrt{42.84}$

In ΔABC, $\text{BC}^2 = 16.8^2 - 7^2 = 233.24$,

$\text{BC} = \sqrt{233.24}$

$\cos\theta = \frac{7}{16.8} \quad \therefore \theta = 1.141$

Major arc of large circle $= 7(2\pi - 2\theta) = 28.008...$

Major arc of small circle $= 3(2\pi - 2\theta) = 12.003...$

Length of belt $= 2\text{BC} + 2\text{CD} + 28.008 + 12.003$

$= 83.6\,\text{cm}$

9

The power of calculus

Answers

Skills check

1 **a** $\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=\dfrac{1-\dfrac{\sin^2\theta}{\cos^2\theta}}{1+\dfrac{\sin^2\theta}{\cos^2\theta}}=\dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}=\cos^2\theta-\sin^2\theta$
$=\cos^2\theta$

$\therefore\ \cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

b $\tan 2\theta=\tan(\theta+\theta)=\dfrac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}$

$=\dfrac{2\tan\theta}{1-\tan^2\theta}$

$\therefore\ \tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

2 **a** $f(x)=3e^{2x}-2x^2,\quad f'(x)=6e^{2x}-4x$

b $g(x)=(x+1)\ln(x^2+2x+1)$

$=(x+1)\ln(x+1)^2=2(x+1)\ln(x+1)$

$g'(x)=\dfrac{2(x+1)}{(x+1)}+2\ln(x+1)=2+2\ln(x+1)$

c $h(x)=\dfrac{e^{x^2}}{x+1}\qquad h'(x)=\dfrac{(x+1)2xe^{x^2}-e^{x^2}}{(x+1)^2}$

$=\dfrac{e^{x^2}(2x^2+2x-1)}{(x+1)^2}$

Exercise 9A

Proofs using differentiation from first principles

Exercise 9B

1 **a** $y=\cot x\qquad \dfrac{dy}{dx}=-\csc^2 x$

b $y=\csc x\qquad \dfrac{dy}{dx}=-\csc x\cot x$

c $y=\sin 3x\qquad \dfrac{dy}{dx}=3\cos 3x$

d $y=\tan(5x-3)\qquad \dfrac{dy}{dx}=5\sec^2(5x-3)$

e $y=\cos(8-3x)\qquad \dfrac{dy}{dx}=3\sin(8-3x)$

f $y=\csc\left(\dfrac{x-3}{4}\right)\qquad \dfrac{dy}{dx}=-\dfrac{1}{4}\csc\left(\dfrac{x-3}{4}\right)\cot\left(\dfrac{x-3}{4}\right)$

g $y=\cot\left(\dfrac{7-2x}{13}\right)\qquad \dfrac{dy}{dx}=\dfrac{2}{13}\csc^2\left(\dfrac{7-2x}{13}\right)$

2 **a** $y=\sin(x^5-3)\qquad \dfrac{dy}{dx}=5x^4\cos(x^5-3)$

b $y=\cos(e^x)\qquad \dfrac{dy}{dx}=-e^x\sin(e^x)$

c $y=\csc(x^2+11)\quad \dfrac{dy}{dx}=-2x\csc(x^2+11)\cot(x^2+11)$

d $y=\cot(4x^3-2x^2+7x+17)$

$\dfrac{dy}{dx}=-(12x^2-4x+7)\csc^2(4x^3-2x^2+7x+17)$

e $y=\tan(\ln(2x+1)),\quad \dfrac{dy}{dx}=\dfrac{2}{2x+1}\sec^2(\ln(2x+1))$

f $y=\sec\left(\sqrt{e^x+1}\right)$

$\dfrac{dy}{dx}=\dfrac{1}{2}e^x(e^x+1)^{-\frac{1}{2}}\sec\left(\sqrt{e^x+1}\right)\tan\left(\sqrt{e^x+1}\right)$

$=\dfrac{e^x\sec\left(\sqrt{e^x+1}\right)\tan\left(\sqrt{e^x+1}\right)}{2\left(\sqrt{e^x+1}\right)}$

$=\dfrac{e^x\sin\left(\sqrt{e^x+1}\right)\sec^2\left(\sqrt{e^x+1}\right)}{2\left(\sqrt{e^x+1}\right)}$

g $y=\sin(\cos(\tan x))$

$\dfrac{dy}{dx}=-\sec^2 x\sin(\tan x)\cos(\cos(\tan x))$

Exercise 9C

1 **a** $y=(2x-1)\cos x$

$\dfrac{dy}{dx}=2\cos x-(2x-1)\sin x$

b $y=(3x-x^2)\sin 2x$

$\dfrac{dy}{dx}=(3-2x)\sin 2x+2(3x-x^2)\cos 2x$

c $y=e^{1-x}\tan x$

$\dfrac{dy}{dx}=e^{1-x}\sec^2 x-e^{1-x}\tan x=e^{1-x}(\sec^2 x-\tan x)$

d $y=\dfrac{\sin x}{x},\quad \dfrac{dy}{dx}=\dfrac{x\cos x-\sin x}{x^2}$

e $y=\dfrac{2x+3}{\sin 2x},\quad \dfrac{dy}{dx}=\dfrac{2\sin 2x-2(2x+3)\cos 2x}{\sin^2 2x}$

f $y=\dfrac{\tan x}{\sqrt{2-x}}$

$\dfrac{dy}{dx}=\left(\sqrt{2-x}\sec^2 x+\dfrac{1}{2}(2-x)^{-\frac{1}{2}}\tan x\times\dfrac{1}{(2-x)}\right)$

$\dfrac{dy}{dx}=\dfrac{2(2-x)\sec^2 x+\tan x}{2(2-x)^{\frac{3}{2}}}$

2 **a** $y = \sin 2x \qquad x = \frac{\pi}{6}$

$$\frac{dy}{dx} = 2\cos 2x = 2\cos\frac{\pi}{3} = 1$$

b $y = \cos 3x \qquad x = \frac{7\pi}{12}$

$$\frac{dy}{dx} = -3\sin 3x = -3\sin\frac{7\pi}{4} = \frac{3}{\sqrt{2}}$$

c $y = \tan(-x) = -\tan x \qquad x = \frac{5\pi}{4}$

$$\frac{dy}{dx} = -\sec^2 x = -\sec^2\frac{5\pi}{4} = -2$$

d $y = (x - 2)\sin x \qquad x = 0$

$$\frac{dy}{dx} = \sin x + (x - 2)\cos x$$
$$= \sin 0 + (-2)\cos 0 = -2$$

e $y = -3x\cos x \qquad x = \frac{\pi}{2}$

$$\frac{dy}{dx} = -3\cos x + 3x\sin x$$
$$= -3\cos\frac{\pi}{2} + 3\frac{\pi}{2}\sin\frac{\pi}{2} = \frac{3\pi}{2}$$

f $y = x^2\tan x \quad x = \frac{3x}{4}$

$$\frac{dy}{dx} = 2x\tan x + x^2\sec^2 x$$
$$= \frac{3\pi}{2}\tan\left(\frac{3\pi}{4}\right) + \frac{9\pi^2}{16}\sec^2\frac{3\pi}{4}$$
$$= -\frac{3\pi}{2} + \frac{9\pi^2}{8}$$

g $y = e^x\sec x \quad x = 0 \quad \frac{dy}{dx} = e^x\sec x\tan x + e^x\sec x$
$= \sec 0\tan 0 + \sec 0 = 1$

3 **a** $y = \sin^2\alpha + \cos^2\alpha = 1 \qquad \frac{dy}{d\alpha} = 0$

b $y = \frac{\tan\beta}{\sin\beta} = \sec\beta \qquad \frac{dy}{d\beta} = \sec\beta\tan\beta$

c $y = \frac{2\tan 2\theta}{1 - \tan^2\theta} = \tan 4\theta \qquad \frac{dy}{d\theta} = 4\sec^2 4\theta$

d $y = \frac{\sin\rho + \sin 2\rho}{\cos\rho + \cos 2\rho} = \frac{2\sin\frac{3\rho}{2}\cos\frac{\rho}{2}}{2\cos\frac{3\rho}{2}\cos\frac{\rho}{2}}$

$y = \tan\frac{3\rho}{2} \qquad \frac{dy}{dx} = \frac{3}{2}\sec^2\frac{3\rho}{2}$

e $y = \frac{(\sin\varphi\sin 2\varphi - \cos\varphi)\sec\varphi}{\sin\varphi - \cos\varphi}$

$$= \frac{2\sin^2\varphi - 1}{\sin\varphi - \cos\varphi} = \frac{\sin^2\varphi - \cos^2\varphi}{\sin\varphi - \cos\varphi}$$

$$= \frac{(\sin\varphi - \cos\varphi)(\sin\varphi + \cos\varphi)}{\sin\varphi - \cos\varphi}$$

$y = \sin\varphi + \cos\varphi \quad \frac{dy}{d\varphi} = \cos\varphi - \sin\varphi$

Exercise 9D

1 **a** $y = \arccos x \therefore x = \cos y$

$$\frac{dx}{dy} = -\sin y \quad \therefore \frac{dx}{dy} = -\frac{1}{\sin y}$$
$$= -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}$$

$$\therefore \quad f'(x) = -\frac{1}{\sqrt{1 - x^2}}$$

b $f(x) = \arcsin 3x,\ f'(x) = \frac{3}{\sqrt{1 - 9x^2}}$

c $f(x) = \arctan(2x + 1),\ f'(x) = \frac{2}{1 + (2x + 1)^2}$

$$= \frac{2}{1 + 4x^2 + 4x + 1}$$

$$f'(x) = \frac{1}{2x^2 + 2x + 1}$$

2 **a** $y = 2x\arcsin x$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{1 - x^2}} + 2\arcsin x$$

b $y = \frac{\text{arc}\cos x}{x}$

$$\frac{dy}{dx} = \frac{\frac{-x}{\sqrt{1 - x^2}} - \arccos x}{x^2}$$

$$\frac{dy}{dx} = \frac{-x - \sqrt{1 - x^2}\arccos x}{x^2\sqrt{1 - x^2}}$$

c $y = (2x + 1)\arctan x$

$$\frac{dy}{dx} = 2\arctan x + \frac{2x + 1}{1 + x^2}$$

d $y = \sqrt{1 - x^2}\arcsin x$

$$\frac{dy}{dx} = \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}} - 2x\frac{1}{2}(1 - x^2)^{-\frac{1}{2}}\arcsin x$$

$$\frac{dy}{dx} = 1 - \frac{x\arcsin x}{\sqrt{1 - x^2}}$$

e $y = (4x^2 + 1)\arctan 2x$

$$\frac{dy}{dx} = 8x\arctan 2x + \frac{2(4x^2 + 1)}{1 + 4x^2}$$

$$\frac{dy}{dx} = 8x\arctan 2x + 2$$

3 **a** $\frac{d}{dx}(\arcsin x + \arccos x) = \frac{1}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - x^2}} = 0$

b $\frac{d}{dx}(\arctan x + \arctan(-x)) = \frac{1}{1 + x^2} - \frac{1}{1 + x^2} = 0$

c $\frac{\mathrm{d}}{\mathrm{d}x}\left(2\arctan x-\arcsin\frac{2x}{x^2+1}\right)$

$$=\frac{2}{1+x^2}-\left(\frac{(x^2+1)2-2x(2x)}{(x^2+1)^2}\right)\frac{1}{\sqrt{1-\frac{4x^2}{(x^2+1)^2}}}$$

$$=\frac{2}{1+x^2}-\frac{\left(2-2x^2\right)}{\left(x^2+1\right)^2}\frac{\left(x^2+1\right)}{\sqrt{\left(x^2+1\right)^2-4x^2}}$$

$$=\frac{2}{1+x^2}-\frac{2(1-x^2)}{(x^2+1)\sqrt{(x^2-1)^2}}=0$$

4 a $x = \sin y$

$1=\cos y\frac{\mathrm{d}y}{\mathrm{d}x}\quad\therefore\quad\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\cos y}$

b $x + y = \tan y$

$1+\frac{\mathrm{d}y}{\mathrm{d}x}=\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x}$

$\frac{\mathrm{d}y}{\mathrm{d}x}(\sec^2 y-1)=1\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}\tan^2 y=1\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\cot^2 y$

c $x + \sin x = y + \cos y$

$1+\cos x=\frac{\mathrm{d}y}{\mathrm{d}x}(1-\sin y)\quad\therefore\quad\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1+\cos x}{1-\sin y}$

d $\mathrm{e}^{\sin y}=x^2$

$\mathrm{e}^{\sin y}\cos y\frac{\mathrm{d}y}{\mathrm{d}x}=2x\quad\therefore\quad\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x}{\mathrm{e}^{\sin y}\cos y}$

e $\cos y=\frac{x}{y}\Rightarrow y\cos y=x$

$\frac{\mathrm{d}y}{\mathrm{d}x}(\cos y-y\sin y)=1\quad\therefore\quad\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\cos y-y\sin y}$

f $\ln(xy)=\tan 2y\Rightarrow\ln x+\ln y=\tan 2y$

$\frac{1}{x}+\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=2\sec^2 2y\frac{\mathrm{d}y}{\mathrm{d}x}$

$\frac{\mathrm{d}y}{\mathrm{d}x}\left(2\sec^2 2y-\frac{1}{y}\right)=\frac{1}{x}$

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{1}{x}}{2\sec^2 2y-\frac{1}{y}}=\frac{y}{2xy\sec^2 2y-x}$

Exercise 9E

1 a $f(x)=\tan 3x\quad\mathrm{P}(0, 0)$

$f'(x)=3\sec^2 3x\quad f'(0)=3$

$y = 3x$

b $f(x)=\sin(2x)-1\qquad\mathrm{P}\left(\frac{\pi}{3},y\right)$

$y=\sin\left(\frac{2\pi}{3}\right)-1=\frac{\sqrt{3}}{2}-1\quad\mathrm{P}\left(\frac{\pi}{3},\frac{\sqrt{3}}{2}-1\right)$

$f'(x)=2\cos(2x)\quad f'\left(\frac{\pi}{3}\right)=-1$

$y-\frac{\sqrt{3}}{2}+1=-1\left(x-\frac{\pi}{3}\right)$

$y=-x+\frac{\pi}{3}+\frac{\sqrt{3}}{2}-1$

c $f(x)=2\cos\left(\frac{x}{2}\right)-\mathrm{e}^{2x}\qquad\mathrm{P}(0, 1)$

$f'(x)=-\sin\left(\frac{x}{2}\right)-2\mathrm{e}^{2x}\qquad\therefore f'(0)=-2$

$y = -2x + 1$

d $f(x)=\ln\left(\tan\left(\frac{x}{3}\right)\right)+2\qquad\mathrm{P}\left(\frac{3\pi}{4},y\right)$

$y=\ln\left(\tan\frac{\pi}{4}\right)+2=2\qquad\mathrm{P}\left(\frac{3\pi}{4},2\right)$

$f'(x)=\frac{\frac{1}{3}\sec^2\left(\frac{x}{3}\right)}{\tan\left(\frac{x}{3}\right)}\qquad f'\left(\frac{3\pi}{4}\right)=\frac{1}{3}\sec^2\left(\frac{\pi}{4}\right)=\frac{2}{3}$

$y-2=\frac{2}{3}\left(x-\frac{3\pi}{4}\right)\qquad y=\frac{2}{3}x-\frac{\pi}{2}+2$

2 a $f(x)=\cos(2x)\qquad\mathrm{P}(0, 1)$

$f'(x)=-2\sin(2x),\qquad f'(0)=0$

equation of normal is $x = 0$

b $f(x)=\tan(4x)\qquad\mathrm{P}\left(\frac{\pi}{16},y\right)$

$y=\tan\frac{\pi}{4}=1\qquad\mathrm{P}\left(\frac{\pi}{16},1\right)$

$f'(x)=4\sec^2(4x),\quad f'\left(\frac{\pi}{16}\right)=4\sec^2\left(\frac{\pi}{4}\right)=8$

$y-1=\frac{-1}{8}\left(x-\frac{\pi}{16}\right)$

$y=\frac{-1}{8}x+\frac{\pi}{128}+1$

c $f(x)=2\mathrm{e}^x\sin\left(\frac{x}{2}\right)\quad\mathrm{P}(0, y)\quad y=0\quad\mathrm{P}(0, 0)$

$f'(x)=\mathrm{e}^x\cos\left(\frac{x}{2}\right)+2\mathrm{e}^x\sin\left(\frac{x}{2}\right)$

$f'(0)=1\qquad y=-x$

d $f(x)=x\cos(2x)-3\quad\mathrm{P}\left(\frac{\pi}{2},y\right)$

$y=\frac{\pi}{2}\cos\pi-3=-\frac{\pi}{2}-3\qquad\mathrm{P}\left(\frac{\pi}{2},-\frac{\pi}{2}-3\right)$

$f'(x)=-2x\sin(2x)+\cos(2x)$

$f'\left(\frac{\pi}{2}\right)=-1\qquad y+\frac{\pi}{2}+3=x-\frac{\pi}{2}$

$y = x - \pi - 3$

3 $\ln(x)=\tan y\quad\mathrm{P}(1, 0)$

$\frac{1}{x}=\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x}\qquad\therefore\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos^2 y}{x}$

At P, $\frac{\mathrm{d}y}{\mathrm{d}x}=1\quad y=x-1$

4 $y + y^2 = \sin 2x \quad P(0, -1)$

$\frac{dy}{dx}(1 + 2y) = 2\cos 2x$

$\frac{dy}{dx} = \frac{2\cos 2x}{1 + 2y} = \frac{2}{-1} = -2$

$y + 1 = \frac{1}{2}x \qquad y = \frac{1}{2}x - 1$

5 $y = \cos(x^2)$

a $(0.974, 0.583)$

b $y = \cos(x^2)$

$\frac{dy}{dx} = -2x\sin(x^2)$

$x = 0.97407123, \frac{dy}{dx} = -1.5833\ldots$

$y - 0.58264678 = -1.5833(x - 0.97407123)$

$y = -1.58x + 2.12$

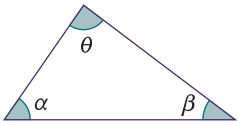
$y = e^{x^2} - 2$

$\frac{dy}{dx} = 2xe^{x^2} = 5.03136\ldots$

$y - 0.58264678 = 5.03136(x - 0.97407123)$

$y = 5.03x - 4.32$

c



$\tan\alpha = 5.03136\ldots$

$\alpha = 1.3746$

$\tan\beta = 1.5833\ldots$

$\beta = 1.00747$

$\theta = \pi - \alpha - \beta = 0.760$ rads

6 $e^y = \sin x + 1 \qquad P(-\pi, 0)$

$e^y\frac{dy}{dx} = -\cos x \quad \frac{dy}{dx} = \frac{-\cos x}{e^y}$

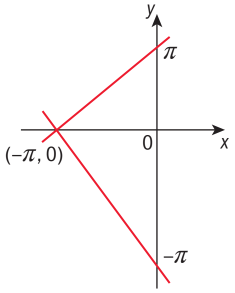
At P, $\frac{dy}{dx} = \frac{1}{e^0} = 1$

Tangent: $y = x + \pi$

Normal: $y = x - \pi$

Area $= \frac{1}{2} \times 2\pi \times \pi$

$= \pi^2$



Exercise 9F

1 **a** $f(x) = \tan x \quad f'(x) = \sec^2 x$

$f''(x) = 2\sec x\sec x\tan x = 2\sec^2 x\tan x$

$f''\left(\frac{\pi}{3}\right) = 2(2)^2\sqrt{3} = 8\sqrt{3}$

b $f(x) = x\sin x, \quad f'(x) = \sin x + x\cos x$

$f''(x) = \cos x + \cos x - x\sin x$

$= 2\cos x - x\sin x$

$f''(0) = 2$

c $f(x) = (x^2 + 1)\cos x$

$f'(x) = 2x\cos x - (x^2 + 1)\sin x$

$f''(x) = 2\cos - 2x\sin x - [+(x^2 + 1)\cos x + 2x\sin x]$

$= 2\cos x - 4x\sin x - (x^2 + 1)\cos x$

$f''(0) = 2 - 1 = 1$

d $f(x) = \sqrt{x}\cos\frac{x}{2}$

$f'(x) = -\frac{1}{2}\sqrt{x}\sin\frac{x}{2} + \frac{1}{2}x^{-\frac{1}{2}}\cos\frac{x}{2}$

$= \frac{-x\sin\frac{x}{2} + \cos\frac{x}{2}}{2\sqrt{x}}$

$$f''(x) = \frac{2\sqrt{x}\left(-\sin\frac{x}{2} - \frac{x}{2}\cos\frac{x}{2} - \frac{1}{2}\sin\frac{x}{2}\right) - \left(-x\sin\frac{x}{2} + \cos\frac{x}{2}\right)x^{\frac{1}{2}}}{4x}$$

$$f''(1) = \frac{2\left(-\frac{3}{2}\sin\frac{1}{2} - \frac{-1}{2}\cos\frac{1}{2}\right) - \left(-\sin\frac{1}{2} + \cos\frac{1}{2}\right)}{4}$$

$$= \frac{-3\sin\frac{1}{2} - \cos\frac{1}{2} + \sin\frac{1}{2} - \cos\frac{1}{2}}{4}$$

$$= \frac{1}{4}\left(-2\sin\frac{1}{2} - 2\cos\frac{1}{2}\right)$$

$$= -\frac{1}{2}\left(\sin\frac{1}{2} + \cos\frac{1}{2}\right)$$

e $f(x) = e^x\sin 2x \quad f'(x) = e^x(2\cos 2x + \sin 2x)$

$f''(x) = e^x(-4\sin 2x + 2\cos 2x + 2\cos 2x + \sin 2x)$

$= e^x(4\cos 2x - 3\sin 2x)$

$f''\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{4}}(0 - 3) = -3e^{\frac{\pi}{4}}$

f $f(x) = 2x\sec x, \quad f'(x) = 2\sec x + 2x\sec x\tan x$

$= 2\sec x(1 + x\tan x)$

$f''(x) = 2\sec x(\tan x + x\sec^2 x)$

$+ 2\sec x\tan x(1 + x\tan x)$

$f''(\pi) = -2(0 + \pi) + 2(-1)(0) = -2\pi$

2 **a** $f(x) = \cos x \quad f'(x) = -\sin x$

$f''(x) = -\cos x \quad f^{(3)}(x) = \sin x$

$$f^{(n)}(x) = \begin{cases} -\sin x, & n = 4k - 3 \\ -\cos x, & n = 4k - 2 \\ \sin x, & n = 4k - 1 \\ \cos x, & n = 4k \end{cases} \qquad k \in \mathbb{Z}^+$$

b $g(x) = \sin 3x \quad g'(x) = 3\cos 3x$

$g''(x) = -9\sin 3x \quad g^{(3)}(x) = -27\cos 3x$

$$g^{(n)}(x) = \begin{cases} 3^n \cos 3x & n = 4k-3 \\ -3^n \sin 3x & n = 4k-2 \\ -3^n \cos 3x & n = 4k-1 \\ 3^n \sin 3x & n = 4k \end{cases} \qquad k \in \mathbb{Z}^+$$

c $h(x) = \cos(ax+b) \quad h'(x) = -a\sin(ax+b)$

$h''(x) = -a^2\cos(ax+b)$

$h^{(3)}(x) = a^3\sin(ax+b)$

$$h^{(n)}(x) = \begin{cases} -a^n \sin(ax+b) & n = 4k-3 \\ -a^n \cos(ax+b) & n = 4k-2 \\ a^n \sin(ax+b) & n = 4k-1 \\ a^n \cos(ax+b) & n = 4k \end{cases} \qquad k \in \mathbb{Z}^+$$

3 $f(x) = \sin 2x \quad a_n = f^{(n-1)}\left(\frac{\pi}{8}\right) \quad n = 1, 2, 3, \ldots$

a $a_1 = f\left(\frac{\pi}{8}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(x) = 2\cos 2x \quad a_2 = f'\left(\frac{\pi}{8}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$

$f''(x) = -4\sin 2x \quad a_3 = f''\left(\frac{\pi}{8}\right) = -\frac{4}{\sqrt{2}} = -2\sqrt{2}$

$f^{(3)}(x) = -8\cos 2x \quad a_4 = f^{(3)}\left(\frac{\pi}{8}\right) = -\frac{8}{\sqrt{2}} = -4\sqrt{2}$

$\frac{1}{\sqrt{2}}, \sqrt{2}, -2\sqrt{2}, -4\sqrt{2}$

b $\frac{1}{\sqrt{2}}(1 + 2 - 4 - 8 + 16 + 32 - 64$

$- 128 + 256 + 512)$

$= \frac{615}{\sqrt{2}}$ or $\frac{615\sqrt{2}}{\sqrt{2}}$

4 **a** P(n): $f(x) = \sin x \Rightarrow f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$,

$n = 0, 1, 2, \ldots$

P(0): $f(x) = \sin x$

Assume P(k): $f^{(k)}(x) = \sin\left(x + \frac{k\pi}{2}\right)$

Prove P(k + 1) $f^{(k+1)}(x) = \cos\left(x + \frac{k\pi}{2}\right)$

$= \sin\left(x + \frac{k\pi}{2} + \frac{\pi}{2}\right)$

$= \sin\left(x + (k+1)\frac{\pi}{2}\right)$

∴ P(k) ⇒ P(k + 1) and P(0) is true

∴ by induction,

$f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right), n = 0, 1, 2, \ldots.$

b P(n): $g(x) = \cos x \Rightarrow g^{(n)}(x) = \sin\left(x + \frac{(n+1)\pi}{2}\right)$,

$n = 0, 1, 2, \ldots$

P(0): $g(x) = \sin\left(x + \frac{\pi}{2}\right) = \cos x$

Assume P(k): $g^{(k)}(x) = \sin\left(x + \frac{(k+1)\pi}{2}\right)$

Prove P(k + 1) $g^{(k+1)}(x) = \cos\left(x + \frac{(k+1)\pi}{2}\right)$

$= \sin\left(x + \frac{(k+1)\pi}{2} + \frac{\pi}{2}\right)$

$= \sin\left(x + \frac{(k+2)\pi}{2}\right)$

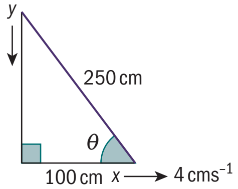
∴ P(k) ⇒ P(k + 1) and P(0) is true

∴ by induction,

$g^{(n)}(x) = \sin\left(x + \frac{(n+1)\pi}{2}\right), n = 0, 1, 2, \ldots.$

Exercise 9G

1



$\frac{dx}{dt} = 4 \quad \cos\theta = \frac{x}{250}$

$-\sin\theta\frac{d\theta}{dt} = \frac{1}{250}\frac{dx}{dt}$

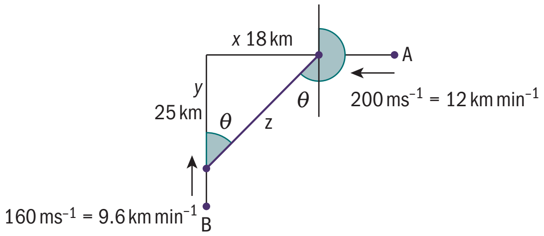
$x = 100 \Rightarrow \cos\theta = \frac{100}{250} = \frac{2}{5}$

$\sin\theta = \sqrt{1 - \frac{4}{25}} = \frac{\sqrt{21}}{5}$

$-\frac{\sqrt{21}}{5}\frac{d\theta}{dt} = \frac{4}{250} \quad \therefore \quad \frac{d\theta}{dt} = -0.0175\,\text{cs}^{-1}$

the angle is decreasing at a rate of 0.0175 cs^{-1}

2



a $x = 18 - 12t \Rightarrow \frac{dx}{dt} = -12$

$y = 25 - 9.6t \Rightarrow \frac{dy}{dt} = -9.6$

$z^2 = x^2 + y^2 \quad \therefore 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$t = 0.5 \Rightarrow x = 12, y = 20.2,$

$z = \sqrt{552.04} = 23.4955\ldots$

$23.4955\frac{dz}{dt} = 12(-12) + 20.2(-9.6)$

$\therefore \frac{dz}{dt} = -14.4 \text{ km min}^{-1} \approx -240 \text{ ms}^{-1}$

Approaching each other at 240 ms^{-1}

b bearing = $\pi + \theta$

$\therefore$ rate of change of bearing = $\frac{d\theta}{dt}$

$$\tan\theta = \frac{x}{y} \quad \Rightarrow \quad \sec^2\theta \frac{d\theta}{dt} = \left(y\frac{dx}{dt} - x\frac{dy}{dt}\right)y^2$$

$t = 1 \Rightarrow x = 6,\ y = 15.4,\ \tan\theta = \frac{6}{15.4}$

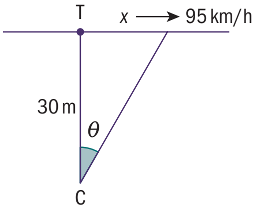
$$\sec^2\theta = 1 + \left(\frac{6}{15.4}\right)^2 = \frac{6829}{5929}$$

$$\frac{6829}{5929}\frac{d\theta}{dt} = \frac{15.4(-12) - 6(-9.6)}{15.4^2}$$

$$\frac{d\theta}{dt} = -0.466 \text{ c min}^{-1}$$

bearing is decreasing at a rate of 0.466 c min^{-1}

3 **a**



$$\frac{dx}{dt} = 95000 \text{ mh}^{-1}$$

$$\tan\theta = \frac{x}{30}$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{30}\frac{dx}{dt}$$

$x = 0,\ \theta = 0 \Rightarrow \sec^2\theta = 1 \Rightarrow \frac{d\theta}{dt} = \frac{95000}{30}$

$= 31.667 \text{ ch}^{-1}$

$= 52.8 \text{ c min}^{-1}$ or 0.880 cs^{-1}

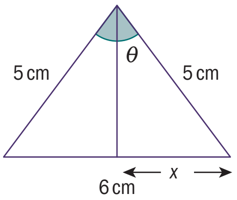
b After 1 sec, $x = \frac{95000}{3600} = 26.338 \ldots$ m

$\tan\theta = \frac{26.388\ldots}{30} = 0.8796\ldots \quad \sec^2\theta = 1.7737 \ldots$

$$1.7737\ldots\frac{d\theta}{dt} = \frac{95000}{30}$$

$\therefore \frac{d\theta}{dt} = 1785.3$ c/h = 29.8 c min^{-1} or 0.496 cs^{-1}

4 **a**



$$\frac{dx}{dt} = -0.05 \text{ cms}^{-1}$$

$$\frac{d}{dt}(2\theta) = 2\frac{d\theta}{dt}$$

$$\sin\theta = \frac{x}{5}$$

$$\cos\theta\frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$$

At $t = 0$, $\cos\theta = \frac{4}{5}$

$$\therefore \frac{d\theta}{dt} = \frac{1}{5} \times -0.05$$

$$\therefore \frac{d\theta}{dt} = -0.0125$$

$\therefore$ angle is decreasing at a rate of 0.0125 cs^{-1}

b $\theta = 30°$ when equilateral

$$\therefore \cos 30°\frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt} = -0.01$$

$$\therefore \frac{d\theta}{dt} = -0.115$$

$\therefore$ angle 2θ is decreasing at 0.0231 cs^{-1}

5 **a** $\frac{dv}{dt} = -2 \text{ cm}^3 \text{ min}^{-1}$

$$v = \frac{4}{3}\pi r^3 \quad \frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$r = 12, \quad -2 = 4\pi(12)^2\frac{dr}{dt} \quad \frac{dr}{dt} = -\frac{1}{288\pi}\text{cm min}^{-1}$$

$$= -0.00111 \text{ cm min}^{-1}$$

radius is decreasing at 0.00111 cm min^{-1}

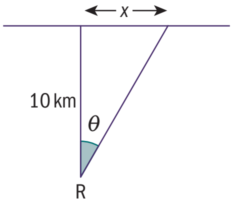
b $A = 4\pi r^2 \quad \frac{dA}{dt} = 8\pi r\frac{dr}{dt}$

$$r = 4, \quad -2 = 4\pi(4)^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{-1}{32\pi}\text{ cm min}^{-1}$$

$$\frac{dA}{dt} = 8\pi(4)\left(\frac{-1}{32\pi}\right) = -1 \text{ cm}^2 \text{ min}^{-1},$$

decreasing at 1 $\text{cm}^2 \text{ min}^{-1}$,

6 **a**



$$\frac{dx}{dt} = 1025 \text{ km h}^{-1}$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{dx}{dt}$$

$$x = 8,\ \tan\theta = \frac{4}{5}\ \sec^2\theta = 1 + \frac{16}{25} = \frac{41}{25}$$

$$\frac{41}{25}\frac{d\theta}{dt} = 102.5 \Rightarrow \frac{d\theta}{dt} = 62.5 \text{ ch}^{-1}$$

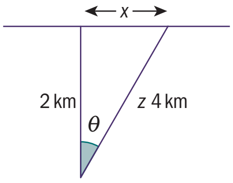
$$\frac{d\theta}{dt} = 0.01761 \text{ cs}^{-1}$$

$= 0.995 \text{ deg s}^{-1}$

b $x = 0,\ \theta = 0,\ \sec^2\theta = 1$

$\frac{d\theta}{dt} = 102.5 \text{ ch}^{-1} = 0.028472 \ldots \text{cs}^{-1} = 1.63 \text{ deg s}^{-1}$

7 **a**



$$\frac{dx}{dt} = 75 \text{ km h}^{-1}$$

$$z^2 = 4 + x^2$$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt}$$

when $z = 4$, $16 = 4 + x^2 \quad \therefore \quad x = \sqrt{12}$

$$4\frac{dz}{dt} = \sqrt{12}(75)$$

$$\frac{dz}{dt} = \frac{75\sqrt{3}}{2} = 65.0 \text{ km h}^{-1}$$

b $\tan\theta = \frac{x}{2} \quad \sec^2\theta\frac{d\theta}{dt} = \frac{1}{2}\frac{dx}{dt}$

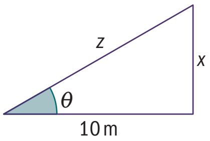
$x = \sqrt{12}, \ \tan\theta = \sqrt{3}, \quad \sec^2\theta = 1 + \left(\sqrt{3}\right)^2 = 4$

$4\frac{d\theta}{dt} = \frac{75}{2}$

$\frac{d\theta}{dt} = \frac{75}{8} = 9.375 \text{ c/h} = 0.00260 \text{ c sec}^{-1}$

$= 0.1 \text{ deg s}^{-1}$ (nearest tenth)

8 **a**



$\frac{dz}{dt} = 5 \text{ ms}^{-1}$

$\cos\theta = \frac{10}{z}$

$-\sin\theta\frac{d\theta}{dt} = \frac{-10}{z^2}\frac{dz}{dt}$

$z = 20, \ \cos\theta = \frac{10}{20} = \frac{1}{2}, \ \sin^2\theta = 1 - \frac{1}{4} = \frac{3}{4},$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\frac{\sqrt{3}}{2}\frac{d\theta}{dt} = \frac{10}{20^2}(5) \Rightarrow \frac{d\theta}{dt} = \frac{1}{4\sqrt{3}} = 0.144 \text{ cs}^{-1}$

or 8.27 deg s^{-1}

b $z^2 = 100 + x^2$

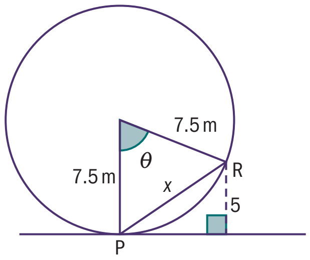
$2z\frac{dz}{dt} = 2x\frac{dx}{dt}$

$z = 20, \ x^2 = 300 \Rightarrow x = 10\sqrt{3}$

$20(5) = 10\sqrt{3}\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{10}{\sqrt{3}} = 5.77 \text{ ms}^{-1}$

9



$\frac{d\theta}{dt} = \frac{4\pi}{60} = \frac{\pi}{15}\text{cs}^{-1}$

$x^2 = 7.5^2 + 7.5^2 - 2 \times 7.5^2 \times \cos\theta$

$x^2 = 112.5 - 112.5\cos\theta$

$2x\frac{dx}{dt} = 112.5\sin\theta\frac{d\theta}{dt}$

When height is 5 m, $\cos\theta = \frac{2.5}{7.5} = \frac{1}{3},$

$\sin^2\theta = 1 - \frac{1}{9} = \frac{8}{9}$

$\therefore \ \sin\theta = \frac{2\sqrt{2}}{3}$

$x^2 = 112.5 - 112.5\left(\frac{1}{3}\right) = 75 \quad \therefore x = 5\sqrt{3}$

$10\sqrt{3}\frac{dx}{dt} = 112.5\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{\pi}{15}\right)$

$\therefore \ \frac{dx}{dt} = \frac{\pi\sqrt{2}}{2\sqrt{3}} = 1.28 \text{ ms}^{-1}$

Exercise 9H

1 **a** $\int \sin 3x \, dx = -\frac{1}{3}\cos 3x + c$

b $\int \cos(2x+1) \, dx = \frac{1}{2}\sin(2x+1) + c$

c $\int \sec^2 3x \, dx = \frac{1}{3}\tan 3x + c$

d $\int \sec^2(1 - x) \, dx = -\tan(1 - x) + c$

e $\int \sin\left(\frac{5x-1}{3}\right)dx = -\frac{3}{5}\cos\left(\frac{5x-1}{3}\right) + c$

f $\int \cos\left(\frac{3x+2}{7}\right)dx = \frac{7}{3}\sin\left(\frac{3x+2}{7}\right) + c$

2 **a** $\int (1 - 2\cos^2 x) \, dx = \int -\cos 2x \, dx = -\frac{1}{2}\sin 2x + c$

b $\int (1 + \tan^2 x) \, dx = \int \sec^2 x \, dx = \tan x + c$

c $\cos 2x = 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\int \sin^2 x \, dx = \frac{1}{2}\int (1 - \cos 2x) \, dx$

$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

d $\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\therefore \ \int \cos^2 x \, dx = \frac{1}{2}\int (1 + \cos 2x) \, dx$

$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + c$

$= \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

e $\int (1 - 2\sin^2(2x))dx = \int \cos 4x \, dx = \frac{1}{4}\sin 4x + c$

f $\int (2 + 2\tan^2(5x)) \, dx = \int 2\sec^2(5x) \, dx$

$= \frac{2}{5}\tan 5x + c$

g $\int (1 + \tan^2 x)(1 - \sin^2 x) \, dx = \int \sec^2 x(\cos^2 x) \, dx$

$= \int 1 \, dx = x + c$

h $\int 4\sin^2 x\cos^2 x \, dx = \int \sin^2(2x) \, dx$

$= \frac{1}{2}\int (1 - \cos 4x) \, dx$

$= \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + c = \frac{1}{2}x - \frac{1}{8}\sin 4x + c$

Exercise 9I

1 **a** $\int (2\sin x - 3\cos x) \, dx = -2\cos x - 3\sin x + c$

b $\int (x^2 - 7\sin x) \, dx = \frac{1}{3}x^3 + 7\cos x + c$

c $\int\left(4e^{x}-\frac{1}{3}\sec^{2}x\right)dx=4e^{x}-\frac{1}{3}\tan x+c$

d $\int(1-\sqrt{2x}+7\sin 3x)dx=x-\sqrt{2}\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{7}{3}\cos 3x+c$

$=x-\frac{2\sqrt{2}x^{\frac{3}{2}}}{3}-\frac{7}{3}\cos 3x+c$

e $\int\left(\frac{5}{2x}+\sec^{2}\left(\frac{x}{3}\right)\right)dx=\frac{5}{2}\ln|x|+3\tan\frac{x}{3}+c$

f $\int\left(\frac{x}{x+1}-\sin\left(\frac{3x}{4}\right)\right)dx=\int\left(1-\frac{1}{x+1}-\sin\left(\frac{3x}{4}\right)\right)dx$

$=x-\ln|x+1|+\frac{4}{3}\cos\frac{3x}{4}+c$

g $\int\left(2^{x}+5\sin\frac{x}{2}-\cos\frac{2x}{3}\right)dx$

$=\frac{2^{x}}{\ln 2}-10\cos\frac{x}{2}-\frac{3}{2}\sin\frac{2x}{3}+c$

h $\int(3^{-2x}-11\sec^{2}(11x))\,dx$

$=\frac{-3^{-2x}}{2\ln 3}-\tan(11x)+c$

Exercise 9J

1 **a** $f'(x)=5-2\cos x \quad f(0)=0$

$f(x)=5x-2\sin x+c$

$0=c \quad \therefore f(x)=5x-2\sin x$

b $f'(x)=4x-6\sin 2x \quad f(0)=1$

$f(x)=2x^{2}+3\cos 2x+c$

$1=3+c \quad \therefore c=-2 \quad f(x)=2x^{2}+3\cos 2x-2$

c $f'(x)=3\cos x-2\sec^{2}x \quad f\left(\frac{\pi}{6}\right)=\frac{-2\sqrt{3}}{3}$

$f(x)=3\sin x-2\tan x+c$

$\frac{-2\sqrt{3}}{3}=\frac{3}{2}-\frac{2\sqrt{3}}{3}+c \quad \therefore c=\frac{-3}{2}$

$f(x)=3\sin x-2\tan x-\frac{3}{2}$

d $f'(x)=3x^{2}-2e^{x}+\cos 4x \quad f(0)=-5$

$f(x)=x^{3}-2e^{x}+\frac{1}{4}\sin 4x+c$

$-5=-2+c \quad \therefore c=-3$

$f(x)=x^{3}-2e^{x}+\frac{1}{4}\sin 4x-3$

e $f'(x)=\frac{3}{x}+\cos(3x)-4 \quad f(1)=\frac{\sin 3}{3}$

$f(x)=3\ln|x|+\frac{1}{3}\sin(3x)-4x+c$

$\frac{\sin 3}{3}=\left(\frac{1}{3}\right)\sin 3-4+c \quad \therefore c=4$

$f(x)=3\ln|x|+\frac{1}{3}\sin(3x)-4x+4$

f $f'(x)=\frac{7}{3-4x}-8x+4e^{2x-1} \quad f\left(\frac{1}{2}\right)=-1$

$f(x)=\frac{-7}{4}\ln|3-4x|-4x^{2}+2e^{2x-1}+c$

$-1=-1+2+c \quad \therefore c=-2$

$f(x)=\frac{-7}{4}\ln|3-4x|-4x^{2}+2e^{2x-1}-2$

2 **a** $f''(x)=4\sin x \quad f'\left(\frac{\pi}{3}\right)=0,\ f(0)=1$

$f'(x)=-4\cos x+c_{1}$

$0=-2+c_{1} \quad \therefore c_{1}=2$

$f'(x)=-4\cos x+2$

$f(x)=-4\sin x+2x+c_{2}$

$1=c_{2} \quad \therefore f(x)=-4\sin x+2x+1$

b $f''(x)=1+\cos x \quad f'(0)=3, \quad f(1)=-\cos(1)$

$f'(x)=x+\sin x+c_{1}$

$3=c_{1} \quad f'(x)=x+\sin x+3$

$f(x)=\frac{x^{2}}{2}-\cos x+3x+c_{2}$

$-\cos(1)=\frac{1}{2}-\cos(1)+3+c_{2},\ c=-\frac{7}{2}$

$f(x)=\frac{1}{2}x^{2}-\cos x+3x-\frac{7}{2}$

c $f''(x)=e^{1-x}+\sin(1-x), \quad f'(1)=2, \quad f(1)=2$

$f'(x)=-e^{1-x}+\cos(1-x)+c_{1}$

$2=-1+1+c_{1} \quad \therefore c_{1}=2$

$f'(x)=-e^{1-x}+\cos(1-x)+2$

$f(x)=e^{1-x}-\sin(1-x)+2x+c_{2}$

$2=1+2+c_{2} \quad \therefore c_{2}=-1$

$f(x)=e^{1-x}-\sin(1-x)+2x-1$

d $f''(x)=e^{2x}+\sin(2x)+x^{3}-2x+1, \quad f'(0)$
$=2, f(0)=2$

$f'(x)=\frac{1}{2}e^{2x}-\frac{1}{2}\cos(2x)+\frac{1}{4}x^{4}-x^{2}+x+c_{1}$

$2=\frac{1}{2}-\frac{1}{2}+c_{1} \quad \therefore c_{1}=2$

$f'(x)=\frac{1}{2}e^{2x}-\frac{1}{2}\cos(2x)+\frac{1}{4}x^{4}-x^{2}+x+2$

$f(x)=\frac{1}{4}e^{2x}-\frac{1}{4}\sin 2x+\frac{1}{20}x^{5}-\frac{1}{3}x^{3}+\frac{1}{2}x^{2}+2x+c_{2}$

$2=\frac{1}{4}+c_{2} \therefore c_{2}=\frac{7}{4}$

$f(x)=\frac{1}{4}e^{2x}-\frac{1}{4}\sin 2x+\frac{1}{20}x^{5}-\frac{1}{3}x^{3}+\frac{1}{2}x^{2}+2x+\frac{7}{4}$

Exercise 9K

1 **a** $\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}}(2x-\sin x)\,dx=\left[x^{2}+\cos x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{2}}$

$=\left(\frac{\pi^{2}}{4}+0\right)-\left(\frac{\pi^{2}}{9}+\frac{1}{2}\right)=\frac{5\pi^{2}}{36}-\frac{1}{2}$

b $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(5+\cos x)\,dx=\left[5x+\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$=\left(\frac{5\pi}{2}+1\right)-\left(\frac{5\pi}{6}+\frac{1}{2}\right)=\frac{5\pi}{3}+\frac{1}{2}$

c $$\int_0^{\frac{\pi}{4}} (2\sec^2 x + 1)\,dx = \left[2\tan x + x\right]_0^{\frac{\pi}{4}}$$
$$= \left(2 + \frac{\pi}{4}\right) - (0) = 2 + \frac{\pi}{4}$$

d $$\int_0^{\frac{\pi}{3}} (e^x + 2\sin x)\,dx = \left[e^x - 2\cos x\right]_0^{\frac{\pi}{3}}$$
$$= (e^{\frac{\pi}{3}} - 1) - (1 - 2) = e^{\frac{\pi}{3}}$$

e $$\int_{-2\pi}^{2\pi} \left(3^{-x} + \frac{1}{4}\cos\frac{x}{4}\right) dx = \left[\frac{-3^{-x}}{\ln 3} + \sin\frac{x}{4}\right]_{-2\pi}^{2\pi}$$
$$= \left(\frac{-3^{-2\pi}}{\ln 3} + 1\right) - \left(\frac{-3^{2\pi}}{\ln 3} - 1\right) = \frac{3^{2\pi} - 3^{-2\pi}}{\ln 3} + 2$$

f $$\int_0^{\frac{\pi}{2}} \left(\frac{e^{3x}}{3} - \frac{2\sin 2x}{5}\right) dx = \left[\frac{e^{3x}}{9} + \frac{1}{5}\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= \left(\frac{1}{9}e^{\frac{3\pi}{2}} - \frac{1}{5}\right) - \left(\frac{1}{9} + \frac{1}{5}\right) = \frac{1}{9}(e^{\frac{3\pi}{2}} - 1) - \frac{2}{5}$$

g $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(1 - \frac{x}{2} + 2\sin 2x\right) dx = \left[x - \frac{x^2}{4} - \cos 2x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$
$$= \left(\frac{\pi}{4} - \frac{\pi^2}{64} - 0\right) - \left(-\frac{\pi}{4} - \frac{\pi^2}{64} - 0\right) = \frac{\pi}{2}$$

h $$\int_0^{\frac{\pi}{2}} (2^x + 3\cos 6x)\,dx = \left[\frac{2^x}{\ln 2} + \frac{1}{2}\sin 6x\right]_0^{\frac{\pi}{12}}$$
$$= \left(\frac{2^{\frac{\pi}{12}}}{\ln 2} + \frac{1}{2}\right) - \left(\frac{1}{\ln 2} + 0\right) = \frac{2^{\frac{\pi}{12}} - 1}{\ln 2} + \frac{1}{2}$$

i $$\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} (x^2 + 2\sec^2 2x)\,dx = \left[\frac{x^3}{3} + \tan 2x\right]_{-\frac{\pi}{8}}^{\frac{\pi}{8}}$$
$$= \left(\frac{\pi^3}{1536} + 1\right) - \left(-\frac{\pi^3}{1536} - 1\right) = \frac{\pi^3}{768} + 2$$

j $$\int_0^{\pi} (16e^{8x} + 9\sin 3x)\,dx = \left[2e^{8x} - 3\cos 3x\right]_0^{\pi}$$
$$= (2e^{8\pi} + 3) - (2 - 3) = 2e^{8\pi} + 4$$

Exercise 9L

1 **a** $$\int 2x\sin x^2\,dx = -\cos x^2 + c$$

b $$\int 3x^2\sqrt{x^3 + 3}\,dx = \frac{2}{3}(x^3 + 3)^{\frac{3}{2}} + c$$

c $$\int (3 - 4x)e^{1+3x-2x^2}\,dx = e^{1+3x-2x^2} + c$$

d $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$
$$= -\ln|\cos x| + c \text{ or } \ln|\sec x| + c$$

e $$\int 2\cos 2x\, e^{\sin 2x}\,dx = e^{\sin 2x} + c$$

f $$\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx = e^{\sqrt{x}} + c$$

g $$\int 2^x \ln 2 \sin(2^x)\,dx = -\cos(2^x) + c$$

h $$\int \frac{\arcsin x}{\sqrt{1 - x^2}}\,dx = \frac{1}{2}(\arcsin x)^2 + c$$

i $$\int \frac{2\,\mathrm{arc\,tan}\,2x}{1 - 4x^2}\,dx = \frac{1}{2}(\arctan 2x)^2 + c$$

2 **a** $$\int x\cos x^2\,dx = \frac{1}{2}\sin x^2 + c$$

b $$\int x^5\sqrt[3]{x^6 - 1}\,dx = \frac{1}{6}(x^6 - 1)^{\frac{4}{3}}\frac{3}{4} + c$$
$$= \frac{1}{8}(x^6 - 1)^{\frac{4}{3}} + c$$

c $$\int (x + 2)e^{3x^2+12x-7} = \frac{1}{6}e^{3x^2+12x-7} + c$$

d $$\int \frac{\tan(5x + 4)}{5}\,dx = \int \frac{\sin(5x + 4)}{5\cos(5x + 4)}\,dx$$
$$= \frac{-1}{25}\ln|\cos(5x+4)| + c \text{ or } \frac{1}{25}\ln|\sec(5x+4)| + c$$

e $$\int \sin 3x \cdot 3^{\cos 3x}\,dx = \frac{-3^{\cos 3x}}{3\ln 3} + c$$

f $$\int \frac{\sin\sqrt[4]{x}}{\sqrt[4]{x^3}}\,dx = -4\cos\sqrt[4]{x} + c$$

g $$\int 5x\cos(5^x)\,dx = \frac{\sin(5^x)}{\ln 5} + c$$

h $$\int \frac{e^{2x} + e^{-2x}}{e^{-2x} - e^{2x}}\,dx = -\frac{1}{2}\ln\left|e^{-2x} - e^{2x}\right| + c$$

i $$\int \frac{\sqrt{\arctan\frac{x}{3}}}{9 + x^2}\,dx = \frac{2}{9}\left(\arctan\frac{x}{3}\right)^{\frac{3}{2}} + c$$

j $$\int (x^2 + x)\cos\left(x^3 + \frac{3}{2}x^2\right) dx = \frac{1}{3}\sin\left(x^3 + \frac{3}{2}x^2\right) + c$$

k $$\int \frac{\arcsin^2(2x + 1)}{\sqrt{-x - x^2}}\,dx = \frac{1}{3}\arcsin^3(2x + 1) + c$$

Exercise 9M

1 $$\int_0^1 3x^2(x^3 - 1)^4\,dx = \left[\frac{1}{5}(x^3 - 1)^5\right]_0^1 = 0 - \left(\frac{-1}{5}\right) = \frac{1}{5}$$

2 $$\int_0^3 \frac{2x}{x^2 + 1}\,dx = \left[\ln|x^2 + 1|\right]_0^3 = \ln 10 - \ln 1 = \ln 10$$

3 $$\int_0^{\frac{\pi}{6}} \cos x\sqrt{\sin x}\,dx = \left[\frac{2}{3}(\sin x)^{\frac{3}{2}}\right]_0^{\frac{\pi}{6}}$$
$$= \frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} = \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}$$

4 $$\int_1^{e^3} \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^{e^3} = \frac{1}{2}(\ln e^3)^2 - 0 = \frac{9}{2}$$

5 $$\int_0^{\ln 2} \frac{e^x}{e^x + 1}\,dx = \left[\ln(e^x + 1)\right]_0^{\ln 2} = \ln(e^{\ln 2} + 1) - \ln 2 = \ln\frac{3}{2}$$

6 $$\int_0^{\frac{\pi}{6}} 2\tan 2x\,dx = \left[-\ln(\cos 2x)\right]_0^{\frac{\pi}{6}} = -\ln\left(\frac{1}{2}\right) = \ln 2$$

7 $$\int_0^1 (x^2+x)\cos\left(x^3+\frac{3}{2}x^2\right)\mathrm{d}x = \frac{1}{3}\left[\sin\left(x^3+\frac{3}{2}x^2\right)\right]_0^1$$
$$= \frac{1}{3}\sin\frac{5}{2}$$

8 $$\int_0^3 2^x\sqrt{2x+1}\ \mathrm{d}x \quad = \frac{1}{\ln 2}\left[\frac{2}{3}(2^x+1)^{\frac{3}{2}}\right]_0^3$$
$$= \frac{2}{3\ln 2}\left(9^{\frac{3}{2}}-2^{\frac{3}{2}}\right) = \frac{2}{3\ln 2}(27-2\sqrt{2})$$

Exercise 9N

1 $$\int xe^x\,\mathrm{d}x = xe^x - \int e^x\,\mathrm{d}x$$
$$= xe^x - e^x + c$$
$$= e^x(x-1)+c$$

$u = x \quad \frac{\mathrm{d}v}{\mathrm{d}x} = e^x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad v = e^x$

2 $$\int (2x+9)\cos x\,\mathrm{d}x$$
$$= (2x+9)\sin x - \int 2\sin x\,\mathrm{d}x$$
$$= (2x+9)\sin x + 2\cos x + c$$

$u = 2x+9$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 2$

$v = \sin x$

3 $$\int (2-5x)\sin x\,\mathrm{d}x$$
$$= -(2-5x)\cos x - \int 5\cos x\,\mathrm{d}x$$
$$=(5x-2)\cos x - 5\sin x + c$$

$u = 2-5x$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = -5$

$v = -\cos x$

4 $$\int (3x-1)e^{3x}\,\mathrm{d}x$$
$$= (3x-1)\,\frac{1}{3}\,e^{3x} - \int e^{3x}\,\mathrm{d}x$$
$$= \frac{1}{3}\,(3x-1)\,e^{3x} - \frac{1}{3}\,e^{3x} + c$$
$$= \frac{1}{3}e^{3x}(3x-2)+c$$

$u = 3x-1$

$\frac{\mathrm{d}v}{\mathrm{d}x} = e^{3x}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 3$

$v = \frac{1}{3}e^{3x}$

5 $$\int (4x-7)e^{4x-1}\,\mathrm{d}x$$
$$= \frac{1}{4}e^{4x-1}\,(4x-7) - \int e^{4x-1}\,\mathrm{d}x$$
$$= \frac{1}{4}e^{4x-1}\,(4x-7) - \frac{1}{4}e^{4x-1} + c$$
$$= \frac{1}{4}e^{4x-1}\,(4x-8) + c$$
$$= e^{4x-1}(x-2)+c$$

$u = 4x-7$

$\frac{\mathrm{d}v}{\mathrm{d}x} = e^{4x-1}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 4$

$v = \frac{1}{4}e^{4x-1}$

6 $$\int \frac{x+3}{2}\sin(2x+3)\mathrm{d}x$$
$$= -\frac{1}{4}\,(x+3)\cos(2x+3)$$
$$+\int \frac{1}{4}\cos(2x+3)\,\mathrm{d}x$$
$$= -\frac{1}{4}(x+3)\cos(2x+3)+\frac{1}{8}\sin(2x+3)+c$$

$u = \frac{x+3}{2}$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin(2x+3)$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2}$

$v = -\frac{1}{2}\cos(2x+3)$

7 $$\int \frac{3-x}{4}\cos\left(\frac{x}{4}\right)\mathrm{d}x$$
$$= (3-x)\sin\left(\frac{x}{4}\right) + \int \sin\left(\frac{x}{4}\right)\mathrm{d}x$$
$$= (3-x)\sin\left(\frac{x}{4}\right) - 4\cos\left(\frac{x}{4}\right)+c$$

$u = \frac{3-x}{4}$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos\left(\frac{x}{4}\right)$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-1}{4}$

$v = 4\sin\left(\frac{x}{4}\right)$

8 $$\int x2^x\,\mathrm{d}x$$
$$\frac{2^x x}{\ln 2} - \int \frac{2^x}{\ln 2}\mathrm{d}x$$
$$= \frac{2^x x}{\ln 2} - \frac{2^x}{(\ln 2)^2}+c$$
$$= \frac{2^x(x\ln 2-1)}{(\ln 2)^2} + c$$

$u = x \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 2^x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad v = \frac{2^x}{\ln 2}$

9 $$\int (1-x)5^x\,\mathrm{d}x$$
$$= \frac{(1-x)5^x}{\ln 5} + \int \frac{5^x}{\ln 5}\mathrm{d}x$$
$$= \frac{(1-x)5^x}{\ln 5} + \frac{5^x}{(\ln 5)^2}+c$$
$$= \frac{5^x((1-x)\ln 5+1)}{(\ln 5)^2}+c$$

$u = 1-x$

$\frac{\mathrm{d}v}{\mathrm{d}x} = 5^x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = -1$

$v = \frac{5^x}{\ln 5}$

10 $$\int \frac{(2-x)}{7.3^x}\mathrm{d}x = \frac{1}{7}\int (2-x)3^{-x}\,\mathrm{d}x$$
$$= \frac{1}{7}\left[\frac{-3^{-x}(2-x)}{\ln 3} - \int \frac{3^{-x}}{\ln 3}\mathrm{d}x\right]$$
$$= \frac{1}{7}\left[\frac{3^{-x}(x-2)}{\ln 3} + \frac{3^{-x}}{(\ln 3)^2}\right]+c$$
$$= \frac{3^{-x}((x-2)\ln 3+1)}{7(\ln 3)^2}+c$$

$u = 2-x$

$\frac{\mathrm{d}v}{\mathrm{d}x} = 3^{-x}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = -1$

$v = \frac{-3^{-x}}{\ln 3}$

11 $$\int \frac{4x\cdot 3^x}{5^x}\mathrm{d}x = \int 4x\left(\frac{3}{5}\right)^x \mathrm{d}x$$
$$= 4x\left(\frac{3}{5}\right)^x \frac{1}{\ln\left(\frac{3}{5}\right)} - \int 4\left(\frac{3}{5}\right)^x \frac{1}{\ln\left(\frac{3}{5}\right)}\mathrm{d}x$$

$u = 4x \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \left(\frac{3}{5}\right)^x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 4$

$v = \left(\frac{3}{5}\right)^x \frac{1}{\ln\left(\frac{3}{5}\right)}$

$$=\frac{4x\cdot 3^{x}}{5^{x}\ln\left(\frac{3}{5}\right)}-4\left(\frac{3}{5}\right)^{x}\frac{1}{\left(\ln\left(\frac{3}{5}\right)\right)^{2}}+c$$

$$=\frac{4\cdot 3^{x}\left(x\ln\left(\frac{3}{5}\right)-1\right)}{5^{x}\left(\ln\left(\frac{3}{5}\right)\right)^{2}}+c$$

Exercise 9O

1 $\int x\ln x\,\mathrm{d}x=\frac{x^{2}}{2}\ln x-\int\frac{x}{2}\mathrm{d}x$

$$=\frac{x^{2}}{2}\ln x-\frac{x^{2}}{4}+c$$

$u=\ln x$

$\frac{\mathrm{d}v}{\mathrm{d}x}=x$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}\quad v=\frac{x^{2}}{2}$

2 $\int(3x+2)\ln x\,\mathrm{d}x$

$$=\left(\frac{3x^{2}}{2}+2x\right)\ln x-\int\left(\frac{3x}{2}+2\right)\mathrm{d}x$$

$$=\left(\frac{3x^{2}}{2}+2x\right)\ln x-\frac{3x^{2}}{4}-2x+c$$

$u=\ln x$

$\frac{\mathrm{d}v}{\mathrm{d}x}=3x+2$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$v=\frac{3x^{2}}{2}+2x$

3 $\int(1-x)\ln x\,\mathrm{d}x$

$$=\left(x-\frac{x^{2}}{2}\right)\ln x-\int\left(1-\frac{x}{2}\right)\mathrm{d}x$$

$$=\left(x-\frac{x^{2}}{2}\right)\ln x-x+\frac{x^{2}}{4}+c$$

$u=\ln x$

$\frac{\mathrm{d}v}{\mathrm{d}x}=1-x$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$v=x-\frac{x^{2}}{2}$

4 $\int x\ln(4x)\,\mathrm{d}x$

$$=\frac{x^{2}}{2}\ln(4x)-\int\frac{x}{2}\mathrm{d}x$$

$$=\frac{x^{2}}{2}\ln(4x)-\frac{x^{2}}{4}+c$$

$$=\frac{x^{2}}{4}(2\ln(4x)-1)+c$$

$u=\ln(4x)$

$\frac{\mathrm{d}v}{\mathrm{d}x}=x$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$v=\frac{x^{2}}{2}$

5 $\int(3x-2)\ln\left(\frac{x}{5}\right)\mathrm{d}x$

$$=\left(\frac{3x^{2}}{2}-2x\right)\ln\left(\frac{x}{5}\right)-\int\left(\frac{3x}{2}-2\right)\mathrm{d}x$$

$$=\left(\frac{3x^{2}}{2}-2x\right)\ln\left(\frac{x}{5}\right)-\frac{3x^{2}}{4}+2x+c$$

$u=\ln\left(\frac{x}{5}\right)$

$\frac{\mathrm{d}v}{\mathrm{d}x}=3x-2$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$v=\frac{3x^{2}}{2}-2x$

6 $\int(3+4x)\ln(3+4x)\mathrm{d}x$

Let $u=3+4x$, $\frac{\mathrm{d}u}{\mathrm{d}x}=4$

$$\int(3+4x)\ln(3+4x)\,\mathrm{d}x=\int u\ln u\,\frac{1}{4}\mathrm{d}u$$

$=\frac{1}{4}\left[\frac{u^{2}}{2}\ln u-\frac{u^{2}}{4}\right]+c$ (using result from qn1)

$$=\frac{u^{2}}{16}(2\ln u-1)+c$$

$$=\frac{(3+4x)^{2}}{16}(2\ln(3+4x)-1)+c$$

7 Let $t=4-11x$, $\frac{\mathrm{d}t}{\mathrm{d}x}=-11$

$$x=\frac{1}{11}(4-t)$$

$$\therefore\ 5+7x=5+\frac{7}{11}(4-t)=\frac{83}{11}-\frac{7t}{11}$$

$$\int(5+7x)\ln(4-11)\mathrm{d}x=\frac{-1}{11}$$

$$\int\frac{(83-7t)}{11}\ln t\,\mathrm{d}t$$

$$=\frac{1}{121}\int(7t-83)\ln t\cdot\mathrm{d}t$$

$$=\frac{1}{121}\left[\left(\frac{7t^{2}}{2}-83t\right)\ln t-\int\left(\frac{7t}{2}-83\right)\mathrm{d}t\right]$$

$u=\ln t$

$\frac{\mathrm{d}v}{\mathrm{d}t}=7t-83$

$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{1}{t}$

$v=\frac{7t^{2}}{2}-83t$

$$=\frac{1}{121}\left[\left(\frac{7t^{2}}{2}-83t\right)\ln t-\frac{7t^{2}}{4}+83t\right]+c$$

$$=\frac{1}{121}\left[\left(\frac{7}{2}(4-11x)^{2}-83(4-11x)\right)\ln(4-11x)-\frac{7}{4}(4-11x)^{2}+83(4-11x)\right]+c$$

$$=\frac{1}{121}\left[\left(\frac{7}{2}(16-88x+121x^{2})-332+913x\right)\ln(4-11x)-\frac{7}{4}(16-88x+121x^{2})+332-913x\right]+c$$

$$=\frac{1}{121}\left[\left(\frac{847}{2}x^{2}+605x-276)\ln(4-11x)\right)-\frac{847}{4}x^{2}-759x+304\right]+c$$

8 $\int x^{2}\ln x\mathrm{d}x$

$$=\frac{x^{3}}{3}\ln x-\int\frac{x^{2}}{3}\mathrm{d}x$$

$$=\frac{x^{3}}{3}\ln x-\frac{x^{3}}{9}+c$$

$u=\ln x$

$\frac{\mathrm{d}v}{\mathrm{d}x}=x^{2}$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$v=\frac{x^{3}}{3}$

9 $$\int (2 - x + x^2) \ln(3x)\, \mathrm{d}x$$

$$u = \ln(3x) \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 2 - x + x^2$$

$$= \left(2x - \frac{x^2}{2} + \frac{x^3}{3}\right) \ln(3x) - \int \left(2 - \frac{x}{2} + \frac{x^2}{3}\right) \mathrm{d}x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \qquad v = 2x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$= \left(2x - \frac{x^2}{2} + \frac{x^3}{3}\right) \ln(3x) - 2x + \frac{x^2}{4} - \frac{x^3}{9} + c$$

Exercise 9P

1 $$\int \log x\, \mathrm{d}x$$

$$u = \log x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

$$= x\log x - \int \frac{1}{\ln 10} \mathrm{d}x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x \ln 10} \qquad v = x$$

$$= x \log x - \frac{x}{\ln 10} + c$$

2 $$\int \log_a x \mathrm{d}x$$

$$u = \log_a x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

$$= x\log_a x - \int \frac{1}{\ln a} \mathrm{d}x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x \ln a} \qquad v = x$$

$$= x \log_a x - \frac{x}{\ln a} + c$$

3 $$\int \arctan x\, \mathrm{d}x$$

$$u = \arctan x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

$$= x\arctan x - \int \frac{x}{1+x^2} \mathrm{d}x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{1+x^2} \qquad v = x$$

$$= x \arctan x - \frac{1}{2} \ln\left|1 + x^2\right| + c$$

4 $$\int \arccos x\, \mathrm{d}s$$

$$u = \arccos \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

$$= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}} \qquad v = x$$

$$= x \arccos x - \frac{1}{2} \cdot 2(1-x^2)^{\frac{1}{2}} + c$$

$$= x \arccos x - \sqrt{1-x^2} + c$$

5 $$\int 2x \arctan x\, \mathrm{d}x$$

$$u = \arctan x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 2x$$

$$= x^2 \arctan x - \int \frac{x^2}{1+x^2} \mathrm{d}x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{1+x^2} \qquad v = x^2$$

$$= x^2 \arctan x - \int 1 - \frac{1}{1+x^2} \mathrm{d}x$$

$$= x^2 \arctan x - x + \arctan x + c$$

$$= (x^2 + 1) \arctan x - x + c$$

6 $$\int x^2 \arcsin x\, \mathrm{d}x$$

$$u = \arcsin x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = x^2$$

$$= \frac{x^3}{3} \arcsin x - \int \frac{x^3}{3\sqrt{1-x^2}} \mathrm{d}x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}} \qquad v = \frac{x^3}{3}$$

$$\text{Let } t = 1 - x^2 \qquad \frac{\mathrm{d}t}{\mathrm{d}x} = -2x$$

$$\therefore\ x\,\mathrm{d}x = \frac{-1}{2}\mathrm{d}t$$

$$x^2 = 1 - t$$

$$\frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\, \mathrm{d}x = \frac{1}{3}\int \frac{1-t}{t^{\frac{1}{2}}}\left(\frac{-1}{2}\right)\mathrm{d}t = \frac{-1}{6}\int (t^{\frac{-1}{2}} - t^{\frac{1}{2}})\mathrm{d}t$$

$$= \frac{-1}{6}\left(2t^{\frac{1}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right) + c = \frac{1}{3}\left(\frac{1}{3}t^{\frac{3}{2}} - t^{\frac{1}{2}}\right) + c$$

$$= \frac{1}{9}t^{\frac{1}{2}}(t-3) + c = \frac{1}{9}\sqrt{1-x^2}(1-x^2-3) + c$$

$$= \frac{-1}{9}(x^2+2)\sqrt{1-x^2} + c$$

$$\therefore \int x^2 \arcsin x\, \mathrm{d}x = \frac{x^3}{3}\arcsin x + \frac{1}{9}(x^2+2)\sqrt{1-x^2} - c$$

Exercise 9Q

1 $$\int x^2 e^x \mathrm{d}x$$

$$u = x^2 \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = e^x$$

$$= x^2 e^x - \int 2x e^x\, \mathrm{d}x$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 2x \qquad v = e^x$$

$$= x^2 e^x - [2xe^x - \int 2e^x \mathrm{d}x]$$

$$v = 2x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = e^x$$

$$= x^2 e^x - 2xe^x + 2e^x + c$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \qquad v = e^x$$

$$= e^x(x^2 - 2x + 2) + c$$

2 $$\int (x^2 + 1) \sin x\, \mathrm{d}x$$

$$u = x^2 + 1 \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$$

$$= -(x^2 + 1) \cos x + \int 2x\cos x\, \mathrm{d}x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \qquad v = -\cos x$$

$$= -(x^2+1) \cos x + 2x\sin x - \int 2\sin x\, \mathrm{d}x$$

$$u = 2x \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$$

$$= -(x^2+1) \cos x + 2x\sin x + 2\cos x + c$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \qquad v = \sin x$$

$$= (1 - x^2)\cos x + 2x\sin x + c$$

3 $$\int (2x - x^2)\cos x\mathrm{d}x$$

$$u = 2x - x^2 \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$$

$$(2x - x^2)\sin x + \int (2x - 2)\sin x\mathrm{d}x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 - 2x \qquad v = \sin x$$

$$= (2x - x^2)\sin x - (2x - 2)\cos x + \int 2\cos x\mathrm{d}x$$

$$u = 2x - 2$$

$$= (2x - x^2)\sin x - (2x - 2)\cos x + 2\sin x + c$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \qquad v = -\cos x$$

$$= (2x - x^2 + 2)\sin x + (2 - 2x)\cos x + c$$

4 $\int(1 + x - x^2)e^{2x}dx$

$u = 1 + x - x^2$

$\frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 1 - 2x$

$v = \frac{1}{2}e^{2x}$

$$= \frac{1}{2}(1 + x - x^2)e^{2x} + \int(2x - 1)\frac{1}{2}e^{2x}dx$$

$u = 2x - 1$

$\frac{dv}{dx} = \frac{1}{2}e^{2x}$

$\frac{du}{dx} = 2$

$v = \frac{1}{4}e^{2x}$

$$= \frac{1}{2}(1 + x - x^2)e^{2x} + \frac{1}{4}(2x - 1)e^{2x} - \int\frac{1}{2}e^{2x}dx$$

$$= \frac{1}{2}(1 + x - x^2)e^{2x} + \frac{1}{4}(2x - 1)e^{2x} - \frac{1}{4}e^{2x} + c$$

$$= \frac{1}{2}e^{x}\left(1 + x - x^2 + x - \frac{1}{2} - \frac{1}{2}\right) + c$$

$$= \frac{1}{2}e^{x}(2x - x^2) + c$$

5 $\int(2x^2 + x + 3)\cos(2x)\,dx$

$u = 2x^2 + x + 3$

$\frac{dv}{dx} = \cos(2x)$

$\frac{du}{dx} = 4x + 1$

$v = \frac{1}{2}\sin(2x)$

$$= \frac{1}{2}(2x^2 + x + 3)\sin 2x - \frac{1}{2}\int(4x + 1)\sin 2x\,dx$$

$u = 4x + 1$

$\frac{dv}{dx} = \sin 2x$

$\frac{du}{dx} = 4$

$v = -\frac{1}{2}\cos 2x$

$$= \frac{1}{2}(2x^2 + x + 3)\sin 2x - \frac{1}{2}\left[\frac{-1}{2}(4x+1)\cos 2x + \int 2\cos 2x\,dx\right]$$

$$= \frac{1}{2}(2x^2 + x + 3)\sin 2x + \frac{1}{4}(4x + 1)\cos 2x - \frac{1}{2}\sin 2x + c$$

$$= \frac{1}{2}(2x^2 + x + 2)\sin 2x + \frac{1}{4}(4x+1)\cos 2x + c$$

6 $\int x^2\sin(1 - 2x)dx$

$u = x^2$

$\frac{dv}{dx} = \sin(1 - 2x)$

$\frac{du}{dx} = 2x$

$v = -\frac{1}{2}\cos(1 - 2x)$

$$= \frac{1}{2}x^2\cos(1 - 2x) - \int x\cos(1 - 2x)\,dx$$

$u = x$

$\frac{dv}{dx} = \cos(1 - 2x)$

$\frac{du}{dx} = 1$

$v = -\frac{1}{2}\sin(1 - 2x)$

$$= \frac{1}{2}x^2\cos(1 - 2x) - \left[-\frac{1}{2}x\sin(1-2x) + \int\frac{1}{2}\sin(1-2x)\,dx\right]$$

$$= \frac{1}{2}x^2\cos(1-2x) + \frac{1}{2}x\sin(1 - 2x) - \frac{1}{4}\cos(1 - 2x) + c$$

$$= \frac{1}{4}(2x^2 - 1)\cos(1-2x) + \frac{1}{2}x\sin(1-2x) + c$$

7 $\int x^2 3^x dx$

$u = x^2 \quad \frac{dv}{dx} = 3^x$

$\frac{du}{dx} = 2x \quad v = \frac{3^x}{\ln 3}$

$$= \frac{x^2 3^x}{\ln 3} - \int 2x\,.\,\frac{3^x}{\ln 3}dx$$

$u = 2x \quad \frac{dv}{dx} = \frac{3^x}{\ln 3}$

$\frac{du}{dx} = 2 \quad v = \frac{3^x}{(\ln 3)^2}$

$$= \frac{x^2 3^x}{\ln 3} - \left[2x\,.\,\frac{3^x}{(\ln 3)^2} - \int 2\,.\,\frac{3^x}{(\ln 3)^2}dx\right]$$

$$= \frac{x^2 3^x}{\ln 3} - 2x\,.\,\frac{3^x}{(\ln 3)^2} + \frac{2.3^x}{(\ln 3)^3} + c$$

$$= \frac{3^x}{(\ln 3)^3}[x^2(\ln 3)^2 - 2x\ln 3 + 2] + c$$

8 $\int(1 + x^3)e^{\frac{x}{2}}dx$

$u = 1 + x^3 \quad \frac{dv}{dx} = e^{\frac{x}{2}}$

$\frac{du}{dx} = 3x^2 \quad v = 2e^{\frac{x}{2}}$

$$= 2(1 + x^3)e^{\frac{x}{2}} - \int 6x^2 e^{\frac{x}{2}}dx$$

$u = 6x^2 \quad \frac{dv}{dx} = e^{\frac{x}{2}}$

$\frac{du}{dx} = 12x \quad v = 2e^{\frac{x}{2}}$

$$= 2(1 + x^3)e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + \int 24xe^{\frac{x}{2}}dx$$

$u = 24x \quad \frac{dv}{dx} = e^{\frac{x}{2}}$

$\frac{du}{dx} = 24 \quad v = 2e^{\frac{x}{2}}$

$$= 2(1 + x^3)e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - \int 48e^{\frac{x}{2}}dx$$

$$= 2(1 + x^3)e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 96e^{\frac{x}{2}} + c$$

$$= e^{\frac{x}{2}}[2x^3 - 12x^2 + 48x - 94] + c$$

9 $\int(x^3 + x^2)\sin 5x\,dx$

$u = (x^3 + x^2)$

$\frac{dv}{dx} = \sin 5x$

$\frac{du}{dx} = 3x^2 + 2x$

$v = \frac{-1}{5}\cos 5x$

$= \frac{-1}{5}(x^3 + x^2)\cos 5x + \int (3x^2 + 2x)\,\frac{1}{5}\cos 5x\,\mathrm{d}x$ $\quad u = 3x^2 + 2x \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{5}\cos 5x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 6x + 2 \quad v = \frac{1}{25}\sin 5x$

$= \frac{-1}{5}(x^3 + x^2)\cos 5x + \frac{1}{25}(3x^2 + 2x)\sin 5x - \int (6x + 2)\,\frac{1}{25}\sin 5x\,\mathrm{d}x$

$= \frac{-1}{5}(x^3 + x^2)\cos 5x + \frac{1}{25}(3x^2 + 2x)\sin 5x + \frac{1}{125}(6x + 2)\cos 5x - \int \frac{6}{125}\cos 5x\,\mathrm{d}x$ $\quad u = 6x + 2 \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{25}\sin 5x \quad \mathrm{d}u = 6 \quad v = \frac{1}{25}\cos 5x$

$= \frac{-1}{5}(x^3 + x^2)\cos 5x + \frac{1}{25}(3x^2 + 2x)\sin 5x + \frac{1}{125}(6x + 2)\cos 5x - \frac{6}{125}\sin 5x + c$

$= \frac{1}{125}(-25x^3 - 25x^2 + 6x + 2)\cos 5x + \frac{1}{625}(75x^2 + 50x - 6)\sin 5x + c$

10 $\int x^4\cos x\,\mathrm{d}x$ $\quad u = x^4 \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

$= x^4\sin x - \int 4x^3\sin x\,\mathrm{d}x$ $\quad \frac{\mathrm{d}u}{\mathrm{d}x} = 4x^3 \quad v = \sin x$

$= x^4\sin x + 4x^3\cos x - \int 12x^2\cos x\,\mathrm{d}x$

$= x^4\sin x + 4x^3\cos x - 12x^2\sin x + \int 24x\sin x\,\mathrm{d}x$

$= x^4\sin x + 4x^3\cos x - 12x^2\sin x - 24x\cos x + \int 24\cos x\,\mathrm{d}x$

$= x^4\sin x + 4x^3\cos x - 12x^2\sin x - 24x\cos x + 24\sin x + c$

$= (x^4 - 12x^2 + 24)\sin x + (4x^3 - 24x)\cos x + c$

11 $\int x^5\,\mathrm{e}^{2x}x\,\mathrm{d}x$ $\quad u = x^5 \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 5x^4 \quad v = \frac{1}{2}\mathrm{e}^{2x}$

$= \frac{1}{2}x^5\mathrm{e}^{2x} - \int \frac{5}{2}x^4\mathrm{e}^{2x}\mathrm{d}x$ $\quad u = \frac{5}{2}x^4 \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 10x^3 \quad v = \frac{1}{2}\mathrm{e}^{2x}$

$= \frac{1}{2}x^5\mathrm{e}^{2x} - \frac{5}{4}x^4\mathrm{e}^{2x} + \int 5x^3\mathrm{e}^{2x}\mathrm{d}x$ $\quad u = 5x^3 \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 15x^2 \quad v = \frac{1}{2}\mathrm{e}^{2x}$

$= \frac{1}{2}x^5\mathrm{e}^{2x} - \frac{5}{4}x^4\mathrm{e}^{2x} + \frac{5}{2}x^3\mathrm{e}^{2x} - \int \frac{15}{2}x^2\mathrm{e}^{2x}\mathrm{d}x$ $\quad u = \frac{15}{2}x^2 \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 15x \quad v = \frac{1}{2}\mathrm{e}^{2x}$

$= \frac{1}{2}x^5\mathrm{e}^{2x} - \frac{5}{4}x^4\mathrm{e}^{2x} + \frac{5}{2}x^3\mathrm{e}^{2x} - \frac{15}{4}x^2\mathrm{e}^{2x} + \int \frac{15}{2}x\mathrm{e}^{2x}\mathrm{d}x$ $\quad u = \frac{15}{2}x \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{15}{2} \quad v = \frac{1}{2}\mathrm{e}^{2x}$

$= \frac{1}{2}x^5\mathrm{e}^{2x} - \frac{5}{4}x^4\mathrm{e}^{2x} + \frac{5}{2}x^3\mathrm{e}^{2x} - \frac{15}{4}x^2\mathrm{e}^{2x} + \frac{15}{4}x\mathrm{e}^{2x} - \int \frac{15}{4}\mathrm{e}^{2x}$

$= \frac{\mathrm{e}^{2x}}{8}(4x^5 - 10x^4 + 20x^3 - 30x^2 + 30x - 15) + c$

Exercise 9R

1 $\int \sin x\mathrm{e}^x\,\mathrm{d}x$ $\quad u = \sin x \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \quad v = \mathrm{e}^x$

$= \mathrm{e}^x\sin^x - \int \mathrm{e}^x\cos x\,\mathrm{d}x$ $\quad u = \cos x \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \quad v = \mathrm{e}^x$

$= \mathrm{e}^x\sin x - \mathrm{e}^x\cos x - \int \sin x\mathrm{e}^x\,\mathrm{d}x$

$2\int \sin x\mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x(\sin x - \cos x) + c$

$\int \sin x\mathrm{e}^x\mathrm{d}x = \frac{1}{2}\mathrm{e}^x(\sin x - \cos x) + c$

2 $\int \mathrm{e}^{2x}\cos x\,\mathrm{d}x$ $\quad u = \mathrm{e}^{2x} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 2\mathrm{e}^{2x} \quad v = \sin x$

$= \mathrm{e}^{2x}\sin x - \int 2\mathrm{e}^{2x}\sin x\,\mathrm{d}x$ $\quad u = 2\mathrm{e}^{2x} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 4\mathrm{e}^{2x} \quad v = -\cos x$

$= \mathrm{e}^{2x}\sin x + 2\mathrm{e}^{2x}\cos x - \int 4\mathrm{e}^{2x}\cos x\,\mathrm{d}x$

$5\int \mathrm{e}^{2x}\cos x\,\mathrm{d}x = \mathrm{e}^{2x}(\sin x + 2\cos x)$

$\therefore \int \mathrm{e}^{2x}\cos x\,\mathrm{d}x = \frac{1}{5}\mathrm{e}^{2x}(\sin x + 2\cos x) + c$

3 $\int \cos 3x e^{4x}\,dx \qquad u = \cos 3x \qquad \frac{dv}{dx} = e^{4x}$

$$\frac{du}{dx} = -3\sin 3x \quad v = \frac{1}{4}e^{4x}$$

$$\int \cos 3x e^{4x}\,dx = \frac{1}{4}e^{4x}\cos 3x \qquad u = \frac{3}{4}\sin 3x \qquad \frac{dv}{dx} = e^{4x}$$

$$+\int \frac{3}{4}\sin 3x e^{4x}\,dx \qquad \frac{du}{dx} = \frac{9}{4}\cos 3x \quad v = \frac{1}{4}e^{4x}$$

$$= \frac{1}{4}e^{4x}\cos 3x + \frac{3}{16}e^{4x}\sin 3x - \int \frac{9}{16}\cos 3x e^{4x}dx$$

$$\frac{25}{16}\int \cos 3x e^{4x}dx = \frac{e^{4x}}{16}(4\cos 3x + 3\sin 3x)$$

$$\int \cos 3x e^{4x}dx = \frac{1}{25}e^{4x}(4\cos 3x + 3\sin 3x) + c$$

4 $\int \frac{\sin(2x)}{e^x}dx = \int \sin 2x \cdot e^{-x}\,dx u = \sin 2x \qquad \frac{dv}{dx} = e^{-x}$

$$\frac{du}{dx} = 2\cos 2x \quad v = -e^{-x}$$

$$= -e^{-x}\sin 2x + \int 2\cos 2x e^{-x}\,dx \; u = 2\cos 2x \qquad \frac{dv}{dx} = e^{-x}$$

$$\frac{du}{dx} = -4\sin 2x v = -e^{-x}$$

$$= -e^{-x}\sin 2x - 2e^{-x}\cos 2x - \int 4\sin 2x e^{-x}\,dx$$

$$5\int \frac{\sin 2x}{e^x}dx = -e^{-x}(\sin 2x + 2\cos 2x)$$

$$\therefore \int \frac{\sin 2x}{e^x}dx = \frac{-1}{5}e^{-x}(\sin 2x + 2\cos 2x) + c$$

Exercise 9S

1 $\int x\sqrt{x+2}\,dx \qquad u = x + 2 \qquad dx = du$

$$= \int (u-2)u^{\frac{1}{2}}\,du$$

$$= \int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + c$$

$$= \frac{2}{15}u^{\frac{3}{2}}(3u - 10) + c$$

$$= \frac{2}{15}(x+2)^{\frac{3}{2}}(3x-4) + c$$

2 $\int 3x\sqrt{1-2x}\,dx \qquad u = 1 - 2x$

$$x = \frac{1}{2}(1-u) \qquad dx = -\frac{1}{2}du$$

$$= \int \frac{3}{2}(1-u)u^{\frac{1}{2}}\left(\frac{-1}{2}\right)du$$

$$= \frac{-3}{4}\int \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$

$$= \frac{-3}{4}\left(\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right) + c$$

$$= \frac{-3}{60}\left(10u^{\frac{3}{2}} - 6u^{\frac{5}{2}}\right) + c$$

$$= \frac{-u^{\frac{3}{2}}}{10}(5 - 3u) + c$$

$$= \frac{-(1-2x)^{\frac{3}{2}}}{10}(2 + 6x) + c = -\frac{1}{5}(1-2x)^{\frac{3}{2}}(1 + 3x) + c$$

3 $\int 5x^2\sqrt{3+4x}\,dx \qquad u = 3 + 4x$

$$x = \frac{1}{4}(u - 3) \qquad dx = \frac{1}{4}du$$

$$= \int \frac{5}{16}(u^2 - 6u + 9)\frac{u^{\frac{1}{2}}}{4}du$$

$$= \frac{5}{64}\int \left(u^{\frac{5}{2}} - 6u^{\frac{3}{2}} + 9u^{\frac{1}{2}}\right)du$$

$$= \frac{5}{64}\left(\frac{2}{7}u^{\frac{7}{2}} - \frac{12}{5}u^{\frac{5}{2}} + 6u^{\frac{3}{2}}\right) + c$$

$$= \frac{1}{448}\left(10u^{\frac{7}{2}} - 84u^{\frac{5}{2}} + 210u^{\frac{3}{2}}\right) + c$$

$$= \frac{u^{\frac{3}{2}}}{224}(5u^2 - 42u + 105) + c$$

$$= \frac{(3+4x)^{\frac{3}{2}}}{224}(5(9 + 24x + 16x^2) - 42(3 + 4x) + 105) + c$$

$$= \frac{(3+4x)^{\frac{3}{2}}}{224}(24 - 48x + 80x^2) + c$$

4 $x\sqrt[3]{x+3}\,dx$

$$= \int u = x + 3 \; dx = du$$

$$= \int (u-3)u^{\frac{1}{3}}\,du \; (u^{\frac{4}{3}} - 3u^{\frac{1}{3}})du$$

$$= \frac{3}{7}u^{\frac{7}{3}} - \frac{9}{4}u^{\frac{4}{3}} + c$$

$$= \frac{3u^{\frac{4}{3}}}{28}(4u - 21) + c$$

$$= \frac{3}{28}(x+3)^{\frac{4}{3}}(4x - 9) + c$$

5 $\int x^2\sqrt[4]{x+1}\,dx \qquad u = x + 1 \qquad dx = du$

$$x = u - 1$$

$$= \int (u^2 - 2u + 1)u^{\frac{1}{4}}\,du = \int \left(u^{\frac{9}{4}} - 2u^{\frac{5}{4}} + u^{\frac{1}{4}}\right)du$$

$$= \frac{4}{13}u^{\frac{13}{4}} - \frac{8}{9}u^{\frac{9}{4}} + \frac{4}{5}u^{\frac{5}{4}} + c$$

$$= \frac{4u^{\frac{5}{4}}}{585}(45u^2 - 130u + 117) + c$$

$$= \frac{4}{585}(x+1)^{\frac{5}{4}}\ (45u^2 - 130u + 117) + c$$

$$= \frac{4}{585}(x+1)^{\frac{5}{4}}(45x^2 - 40x + 32) + c$$

6 $\int x^3\sqrt[5]{1-}x\ \mathrm{d}x$ $\qquad u = 1 - x \quad \mathrm{d}x = -\mathrm{d}u$

$x = 1 - u$

$$= -\int (1-3u+3u^2-u^3)u^{\frac{1}{5}}\ \mathrm{d}u$$

$$= -\int \left(u^{\frac{1}{5}} - 3u^{\frac{6}{5}} + 3u^{\frac{11}{5}} - u^{\frac{16}{5}}\right)\mathrm{d}u$$

$$= -\frac{5}{6}u^{\frac{6}{5}} + \frac{15}{11}u^{\frac{11}{5}} - \frac{15}{16}u^{\frac{16}{5}} + \frac{5}{21}u^{\frac{21}{5}} + c$$

$$= -\frac{5}{3696}u^{\frac{6}{5}}(616 - 1008u + 693u^2 - 176u^3) + c$$

$$= -\frac{5}{3696}(1-x)^{\frac{6}{5}}(616 - 1008(1-x) + 693(1-2x+x^2)$$
$$-176(1-3x+3x^2-x^3)) + c$$

$$= -\frac{5(1-x)^{\frac{6}{5}}}{3696}(125 + 150x + 165x^2 + 176x^3) + c$$

Exercise 9T

1 $\int \cos^3 x\ \mathrm{d}x = \int (1 - \sin^2 x)\cos x\ \mathrm{d}x$

$$= \int (\cos x - \sin^2 x \cos x)\ \mathrm{d}x$$

$$= \sin x - \frac{1}{3}\sin^3 x + c$$

2 $\int \cos^4 x\ \mathrm{d}x = \int (\cos^2 x)^2\ \mathrm{d}x$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2\ \mathrm{d}x$$

$$= \int \frac{1 + 2\cos 2x + \cos^2 2x}{4}\ \mathrm{d}x$$

$$= \int \left(\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cdot\frac{1+\cos 4x}{2}\right)\mathrm{d}x$$

$$= \int \left(\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right)\mathrm{d}x$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$$

3 $\int \sin^5\left(\frac{x}{5}\right)\mathrm{d}x = \int \left(1 - \cos^2\left(\frac{x}{5}\right)\right)^2 \sin\left(\frac{x}{5}\right)\mathrm{d}x$

$$= \int \left(\sin\frac{x}{5} - 2\cos^2\frac{x}{5}\sin\frac{x}{5} + \cos^4\frac{x}{5}\sin\frac{x}{5}\right)\mathrm{d}x$$

$$= -5\cos\frac{x}{5} + \frac{10}{3}\cos^3\frac{x}{5} - \cos^5\frac{x}{5} + c$$

4 $\int 48\cos^6(2x)\ \mathrm{d}x = \int 48(\cos^2(2x))^3\ \mathrm{d}x$

$$= \int 48\left(\frac{1+\cos 4x}{2}\right)^3\mathrm{d}x$$

$$= \int 6(1 + 3\cos 4x + 3\cos^2 4x + \cos^3 4x)\ \mathrm{d}x$$

$$= \int (6 + 18\cos 4x + 9(1 + \cos 8x)$$
$$+ 6\cos 4x(1 - \sin^2 4x))\ \mathrm{d}x$$

$$= \int (15 + 24\cos 4x + 9\cos 8x - 6\cos 4x \sin^2 4x)\ \mathrm{d}x$$

$$= 15x + 6\sin 4x + \frac{9}{8}\sin 8x - \frac{1}{2}\sin^3 4x + c$$

Exercise 9U

1 $\int \sqrt{4-x^2}\ \mathrm{d}x \quad x = 2\sin\theta \quad \mathrm{d}x = 2\cos\theta\ \mathrm{d}\theta$

$4 - x^2 = 4 - 4\sin^2\theta = 4(1 - \sin^2\theta) = 4\cos^2\theta$

$\sqrt{4-x^2} = 2\cos\theta$

$$\int \sqrt{4-x^2}\ \mathrm{d}x = \int 2\cos\theta.\ 2\cos\theta\ \mathrm{d}\theta = \int 4\cos^2\theta\ \mathrm{d}\theta$$

$$= \int 2(1 + \cos 2\theta)\ \mathrm{d}\theta$$

$= 2\theta + \sin 2\theta + \mathrm{c}$

$= 2\theta + 2\sin\theta\cos\theta + \mathrm{c}$

$\theta = \arcsin\frac{x}{2} \quad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{x^2}{4}} = \frac{\sqrt{4-x^2}}{2}$

$$\int \sqrt{4-x^2}\ \mathrm{d}x = 2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + c$$

2 $\int \frac{1}{\sqrt{x^2-1}}\ \mathrm{d}x \quad x = \sec\theta \quad \mathrm{d}x = \sec\theta\tan\theta\ \mathrm{d}\theta$

$x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$

$\sqrt{x^2-1} = \tan\theta$

$$\int \frac{1}{\sqrt{x^2-1}}\ \mathrm{d}x = \int \frac{1}{\tan\theta}\sec\theta\ \tan\theta\ \mathrm{d}\theta$$

$$= \int \sec\theta\ \mathrm{d}\theta$$

$= \ln \sec\theta + \tan\theta\ |\ + c$

$= \ln\left|x + \sqrt{x^2-1}\right| + c$

3 $\int \sqrt{x^2+9}\ \mathrm{d}x \quad x = 3\tan\theta \quad \mathrm{d}x = 3\sec^2\theta\ \mathrm{d}\theta$

$x^2 + 9 = 9(\tan^2\theta + 1) = 9\sec^2\theta$

$\sqrt{x^2+9} = 3\sec\theta$

$$\int \sqrt{x^2+9}\ \mathrm{d}x = \int 3\sec\theta\ 3\sec^2\theta\ \mathrm{d}\theta$$

$$= \int 9\sec^3\theta = 9\int \sec\theta\sec^2\theta\ \mathrm{d}\theta$$

Using integration by parts:

$u = \sec\theta \qquad \frac{\mathrm{d}v}{\mathrm{d}\theta} = \sec^2\theta$

$\frac{\mathrm{d}u}{\mathrm{d}\theta} = \sec\theta\tan\theta \qquad v = \tan\theta$

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, \mathrm{d}\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta\,(\sec^2\theta - 1)\, \mathrm{d}\theta$$

$$= \sec\theta\tan\theta - \int \sec^2\theta\, \mathrm{d}\theta + \int \sec\theta\, \mathrm{d}\theta$$

$$2\int \sec^3\theta\, \mathrm{d}\theta = \sec\theta\tan\theta + \ln|\ \sec\theta + \tan\theta|$$

$$\therefore \int \sec^3\theta\ \ \mathrm{d}\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\sec\theta + \tan\theta| + c$$

$$\therefore \int \sqrt{x^2+9}\,\mathrm{d}x = \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\ \sec\theta$$

$$+ \tan\theta\ |\ln + c$$

$$\int \sqrt{x^2+9}\,\mathrm{d}x = \frac{1}{2}x\sqrt{x^2+9} + \frac{9}{2}\ln\left|\frac{\sqrt{x^2+9}+x}{3}\right| + c$$

$$= \frac{1}{2}x\sqrt{x^2+9} + \frac{9}{2}\ln\left|\sqrt{x^2+9}+x\right| + k$$

4 $\int \frac{3}{\sqrt{36-x^2}}\,\mathrm{d}x \quad x = 6\sin\theta \qquad \mathrm{d}x = 6\cos\theta\,\mathrm{d}\theta$

$$36 - x^2 = 36\,(1 - \sin^2\theta) = 36\cos^2\theta$$

$$\sqrt{36-x^2} = 6\cos\theta$$

$$\int \frac{3}{\sqrt{36-x^2}}\,\mathrm{d}x = \int \frac{3}{6\cos\theta}\,.\,6\cos\theta\,\mathrm{d}\theta = \int 3\,\mathrm{d}\theta$$

$$= 3\theta + c$$

$$= 3\arcsin\frac{x}{6} + c$$

5 $\int 3\sqrt{x^2-16}\ \mathrm{d}x \quad x = 4\sec\theta \quad \mathrm{d}x = 4\sec\theta\tan\theta\mathrm{d}\theta$

$$x^2 - 16 = 16\,(\sec^2\theta - 1\,) = 16\tan^2\theta$$

$$\sqrt{x^2-16} = 4\tan\theta$$

$$\int 3\sqrt{x^2-16}\ \mathrm{d}x = \int 12\tan\theta.4\sec\theta\tan\theta\ \mathrm{d}\theta$$

$$= \int 48\sec\theta\tan^2\theta\,\mathrm{d}\theta$$

$$= \int 48\sec\theta(\sec^2\theta - 1)\mathrm{d}\theta$$

$$= \int (48\sec^3\theta - 48\sec\theta)\ \mathrm{d}\theta$$

(see qn. 3 for $\int \sec^3\theta$)

$$= 48\left(\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|\right)$$

$$- 48\ln\ |\sec\theta + \tan\theta|\ + \mathrm{c}$$

$$= 24\sec\theta\tan\theta - 24\ln\ |\sec\theta + \tan\theta|\ + c$$

$$= 6x\,\frac{1}{4}\sqrt{x^2-16}\ {-24}\ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| + c$$

$$= \frac{3}{2}x\sqrt{x^2-16} - 24\ln\ |x + \sqrt{x^2-16}| + k$$

6 $\int \frac{5}{\sqrt{x^2+121}}\,\mathrm{d}x \quad x = 11\tan\theta \quad \mathrm{d}x = 11\sec^2\theta\,\mathrm{d}\theta$

$$x^2 + 121 = 121(\tan^2\theta + 1) = 121\sec^2\theta$$

$$\sqrt{x^2+121} = 11\sec\theta$$

$$\int \frac{5}{\sqrt{x^2+121}}\,\mathrm{d}x = \int \frac{5}{11\sec\theta}.\ 11\sec^2\theta\,\mathrm{d}\theta$$

$$= \int 5\sec\theta\,\mathrm{d}\theta$$

$$= 5\ln|\sec\theta + \tan\theta| + c$$

$$= 5\ln\left|\frac{\sqrt{x^2+121}+x}{11}\right| + c$$

$$= 5\ln\left|\sqrt{x^2+121}+x\right| + k$$

7 $\int \frac{2}{\sqrt{81-4x^2}}\,\mathrm{d}x \quad x = \frac{9}{2}\sin\theta \quad \mathrm{d}x = \frac{9}{2}\cos\theta\,\mathrm{d}\theta$

$$81 - 4x^2 = 81 - 81\sin^2\theta = 81\cos^2\theta$$

$$\sqrt{81-4x^2} = 9\cos\theta$$

$$\int \frac{2}{\sqrt{81-4x^2}}\,\mathrm{d}x = \int \frac{2}{9\cos\theta}\,\frac{9}{2}\cos\theta\ \mathrm{d}\theta = \int 1\mathrm{d}\theta$$

$$= \theta + c$$

$$= \arcsin\frac{2x}{9} + c$$

8 $\int \sqrt{3x^2-75}\ \mathrm{d}x = \sqrt{3}\int \sqrt{x^2-25}\ \mathrm{d}x$

$x = 5\sec\theta$

$\mathrm{d}x = 5\sec\theta\tan\theta\,\mathrm{d}\theta$

$x^2 - 25 = 25(\sec^2\theta - 1) = 25\tan^2\theta$

$$\int \sqrt{3x^2-75}\ \mathrm{d}x = \sqrt{3}\int 5\tan\theta\cdot 5\sec\theta\tan\theta\,\mathrm{d}\theta$$

$$= \sqrt{3}\int 25\sec\theta\tan^2\theta\,\mathrm{d}\theta$$

$$= \sqrt{3}\int 25\sec\theta\,(\sec^2\theta - 1)\ \mathrm{d}\theta$$

(see qn 3 for $\int \sec^3\ \mathrm{d}\theta$)

$$= 25\sqrt{3}\begin{bmatrix}\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| \\ -\ln|\sec\theta + \tan\theta|\end{bmatrix} + c$$

$$= \frac{25\sqrt{3}}{2}(\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|) + c$$

$$= \frac{25\sqrt{3}}{2}\left(\frac{x}{s}\,\frac{\sqrt{x^2-25}}{5} - \ln\left|\frac{x+\sqrt{x^2-25}}{5}\right|\right) + c$$

$$= \frac{\sqrt{3}}{2}x\sqrt{x^2-25} - \frac{25\sqrt{3}}{2}\ln\left|x + \sqrt{x^2-25}\right| + k$$

9 $\int \frac{7}{\sqrt{7x^2+28}}\,\mathrm{d}x = \sqrt{7}\int \frac{1}{\sqrt{x^2+4}}\,\mathrm{d}x$

$x = 2\tan\theta \qquad \mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta$

$x^2 + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta$

$\sqrt{x^2+4} = 2\sec\theta$

$$\int \frac{7}{\sqrt{7x^2+28}}\,dx = \sqrt{7}\int \frac{1}{2\sec\theta}2\sec^2\theta\,d\theta$$

$$= \sqrt{7}\int \sec\theta\,d\theta$$

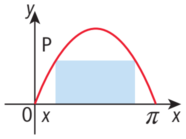
$$= \sqrt{7}\ \ln\ |\sec\theta + \tan\ \theta| + c$$

$$= \sqrt{7}\ \ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| + c$$

$$= \sqrt{7}\ \ln\left|\sqrt{x^2+4}+x\right| + k$$

Exercise 9V

1



P(x, sin$x$)

A = $(\pi - 2x)$ sin$x$

Max. area = 1.12 when $x = 0.71046$

length = $\pi - 2x = 1.72$

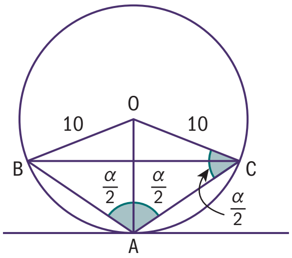
height = sin$x$ = 0.652

2 A (1, –1) P (x, cos$x$)

$\text{AP} = \sqrt{(x-1)^2 + (\cos x + 1)^2}$

Min. distance = 1.11 (at the point (1.78, –0.025))

3 a



In Δ AOC, angle AOC = $\pi - \alpha$

area Δ AOC = $\frac{1}{2}$ .10.10 sin $(\pi - \alpha)$

= 50 sin α

Similarly, area AOB = 50sin α

In Δ BOC, angle BOC = $2\pi - 2\alpha$

area BOC = $\frac{1}{2}$ · 10.10 sin $(2\pi - 2\alpha)$

= –50 sin2α

∴ area ABC = 50 sinα + 50 sinα – (–50sin2α)

A (α) = 100 sinα + 50sin2α

= 100 sinα + 100 sinαcosα

A (α) = 100(1 + cosα) sinα

b For maximum area, $\alpha = 1.05$

4 a $t = 2\sqrt{\frac{15}{g\sin 2\theta}}$ we require t to be a minimum

$\theta = 0.785 = \frac{\pi}{4}$

b $\theta = 0.7854$ $1 = \frac{15}{\cos\theta}$

= 21.213m (nearest mm)

5 a $d\,(t) = \sin\left(\frac{\pi t}{6}\right) + \cos\left(\frac{\pi t}{6}\right)$

$v\,(t) = \frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right) - \frac{\pi}{6}\sin\left(\frac{\pi t}{6}\right)$

$a\,(t) = \frac{-\pi^2}{36}\sin\left(\frac{\pi t}{6}\right)\frac{-\pi^2}{36}\cos\left(\frac{\pi t}{6}\right)$

$\therefore\ a\,(t) = \frac{-\pi^2}{36}d\,(t)$

∴ acceleration is proportional to displacement.

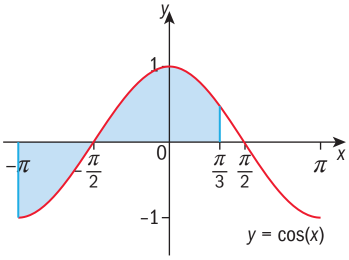
b Max. speed = 0.740 ms^{-1} when t = 4.50 s (velocity in negative and a minimum at this time).

6 Min. height = 1.82 m when x = 1.31 m

∴ the first pole is nearer to the point of minimum height.

Exercise 9W

1 a



$$\int_{-\pi}^{-\frac{\pi}{2}} \cos x\,dx = [\sin x]_{-\pi}^{\frac{-\pi}{2}}$$

$$= \sin\left(\frac{-\pi}{2}\right) - \sin\,(-\pi)$$

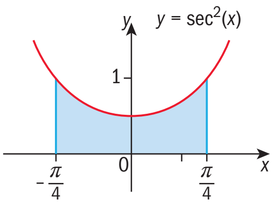
$$= -1 - 0 = -1$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{3}} \cos x\,dx = [\sin\,x]_{-\frac{\pi}{2}}^{\frac{\pi}{3}} = \sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{-\pi}{2}\right)$$

$$= \frac{\sqrt{3}}{2} + 1$$

$\therefore$ total area = $2 + \frac{\sqrt{3}}{2}$

b

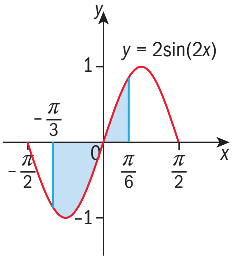


$$\text{Area} = 2\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$$

$$= 2[\tan x]_0^{\frac{\pi}{4}}$$

$$= 2\tan\frac{\pi}{4} = 2$$

c



$$\int_{-\frac{\pi}{2}}^{0} \sin 2x \ \mathrm{d}x = [-\cos 2x]_{-\frac{\pi}{3}}^{0}$$

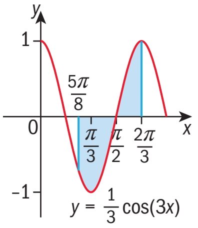
$$= -\cos 0 + \cos\left(-\frac{2\pi}{3}\right) \ = -1 + \left(-\frac{1}{2}\right) = -\frac{3}{2}$$

$$\int_{0}^{\frac{\pi}{6}} 2 \sin 2x \ \mathrm{d}x = [-\cos \ 2x]_{0}^{\frac{\pi}{6}}$$

$$= -\cos\frac{\pi}{3} + \cos 0 \ = -\frac{1}{2} + 1 = \frac{1}{2}$$

area = 2

d



$$\int_{\frac{5\pi}{18}}^{\frac{\pi}{2}} \frac{1}{3}\cos 3x \,\mathrm{d}x = \frac{1}{9}[\sin 3x]_{\frac{5\pi}{18}}^{\frac{\pi}{2}}$$

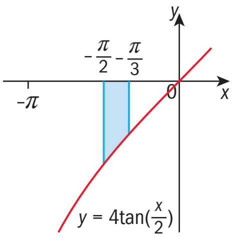
$$= \frac{1}{9}\left(\sin\frac{3\pi}{2} - \sin\frac{5\pi}{6}\right) \ = \frac{1}{9}\left(-1 - \frac{1}{2}\right) = -\frac{1}{6}$$

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{3}\cos 3x \,\mathrm{d}x = \frac{1}{9}[\sin \ 3x]_{\frac{\pi}{2}}^{\frac{2\pi}{3}} = \frac{1}{9}\left(\sin 2\pi - \sin\frac{3\pi}{2}\right)$$

$$= \frac{1}{9}(0 - (-1)) = \frac{1}{9}$$

$$\text{area} = \frac{1}{6} + \frac{1}{9} = \frac{5}{18}$$

e



$$\int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} 4\tan\left(\frac{x}{2}\right) \mathrm{d}x = 8\left[\ln\left|\sec\frac{x}{2}\right|\right]_{-\frac{\pi}{2}}^{-\frac{\pi}{3}}$$

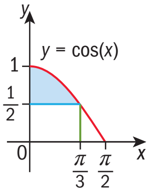
$$= 8\left(\ln\left|\sec\left(-\frac{\pi}{6}\right)\right| - \ln\left|\sec\left(-\frac{\pi}{4}\right)\right|\right)$$

$$= 8\left(\ln\frac{2}{\sqrt{3}} - \ln\left(\sqrt{2}\right)\right)$$

$$= 8\ln\sqrt{\frac{2}{3}} = -8\ln\sqrt{\frac{3}{2}} = -4\ln\left(\frac{3}{2}\right)$$

$$\text{area} = 4 \ \ln\left(\frac{3}{2}\right)$$

2 **a**



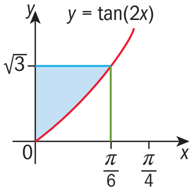
$$\text{Area} = \int_{0}^{\frac{\pi}{3}} \cos x \,\mathrm{d}x - \frac{\pi}{6}$$

$$= [\sin x]_{0}^{\frac{\pi}{3}} - \frac{\pi}{6}$$

$$= \sin\frac{\pi}{3} - \sin 0 - \frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

b



$$\text{Area} = \frac{\pi\sqrt{3}}{6} - \int_{0}^{\frac{\pi}{2}} \tan \ 2x \ \mathrm{d}x$$

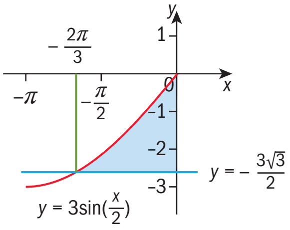
$$= \frac{\pi\sqrt{3}}{6} - \left[\frac{1}{2}\ln|\sec(2x)|\right]_{0}^{\frac{\pi}{6}}$$

$$= \frac{\pi\sqrt{3}}{5} - \frac{1}{2}\left[\ln\left|\sec\frac{\pi}{3}\right| - \ln|\sec 0|\right]$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{1}{2}(\ln 2 - \ln 1)$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{1}{2}\ln 2$$

c



$$3\sin\frac{x}{2} = -3\frac{\sqrt{3}}{2}$$

$$\sin\frac{x}{2} = -\frac{\sqrt{3}}{2}$$

$$\frac{x}{2} = -\frac{\pi}{3}$$

$$x = -\frac{2\pi}{3}$$

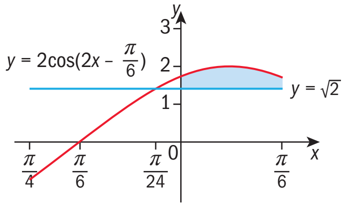
$$\int_{\frac{-2\pi}{3}}^{0} 3\sin\frac{x}{2} \ \mathrm{d}x = -6\left[\cos\left(\frac{x}{2}\right)\right]_{\frac{-2\pi}{3}}^{0}$$

$$= -6\left[\cos \ 0 - \cos\left(-\frac{\pi}{3}\right)\right]$$

$$= -6\left(1 - \frac{1}{2}\right) = -3$$

$$\text{Area} = \frac{2\pi}{3}\left(\frac{3\sqrt{3}}{2}\right) - 3 = \sqrt{3}\pi - 3$$

d



$$2\cos\left(2x-\frac{\pi}{6}\right)=\sqrt{2}$$

$$2x-\frac{\pi}{6}=\frac{\pi}{4}$$

$$2x=\frac{5\pi}{12}$$

$$x=\frac{5\pi}{24}$$

$$\text{Area} = \int_0^{\frac{5\pi}{24}} 2\cos\left(2x-\frac{\pi}{6}\right)\,\mathrm{d}x-\frac{5\pi}{24}(\sqrt{2})$$

$$=\left[\sin\left(2x-\frac{\pi}{6}\right)\right]_0^{\frac{5\pi}{24}}-\frac{5\pi}{24}(\sqrt{2})$$

$$=\sin\left(\frac{\pi}{4}\right)-\sin\left(-\frac{\pi}{6}\right)-\frac{5\pi\sqrt{2}}{24}$$

$$=\frac{\sqrt{2}+1}{2}-\frac{5\pi\sqrt{2}}{24}$$

e $\tan\frac{x}{3}=\frac{1}{\sqrt{3}}$

$$\frac{x}{3}=\frac{\pi}{6}$$

$$x=\frac{\pi}{2}$$

$$\text{Area} = \frac{\pi}{2\sqrt{3}} - \int_0^{\frac{\pi}{2}} \tan\frac{x}{3}\,\mathrm{d}x$$

$$=\frac{\pi}{2\sqrt{3}}-\left[3\ln\left|\sec\frac{x}{6}\right|\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi}{2\sqrt{3}}-3\left(\ln\left|\sec\frac{\pi}{6}\right|-\ln|\sec\ 0|\right)$$

$$=\frac{\pi}{2\sqrt{3}}-3\ln\frac{2}{\sqrt{3}}$$

3 $\text{Area} = \int_0^{2.51327}\left(\cos\frac{x}{2}-\cos 2x\right)\mathrm{d}x = 2.38$

4 $\frac{\mathrm{d}y}{\mathrm{d}x}=\sec^2 x$

$\left(\frac{\pi}{4},1\right)\ \frac{\mathrm{d}y}{\mathrm{d}x}=2$

Tangent: $y-1=2\left(x-\frac{\pi}{4}\right)$

$$y=2x-\frac{\pi}{2}+1$$

if $y=0,\ 2x=\frac{\pi}{2}-1$

$$x=\frac{\pi}{4}-\frac{1}{2}$$

$$\text{Area} = \int_0^{\frac{\pi}{4}}\tan x\,\mathrm{d}x-\frac{1}{2}\left(\frac{1}{2}\right)$$

$$=\left[\ln|\sec x|\right]_0^{\frac{\pi}{4}}-\frac{1}{4}=\ln\left|\sec\frac{\pi}{4}\right|-\ln|\sec\ 0|-\frac{1}{4}$$

$$=\ln\sqrt{2}-\frac{1}{4}=\frac{1}{2}\ln 2-\frac{1}{4}$$

5 $\text{Area} = \int_0^{2.57915}\left|2\sin x-\mathrm{e}^{\frac{x}{2}-4}-1\right|\mathrm{d}x = 1.55$

6 $y=\frac{8}{4+x^2}\quad y=\frac{x^2}{4}$

a $\frac{8}{4+x^2}=\frac{x^2}{4}\Rightarrow 32=4x^2+x^4$

$$x^4+4x^2-32=0$$

$$(x^2+8)(x^2-4)=0$$

$x^2=4,\ x=\pm 2\quad (2, 1), (-2, 1)$

b, c $\text{Area} = \int_{-2}^{2}\left(\frac{8}{4+x^2}-\frac{x^2}{4}\right)\mathrm{d}x$

$$=\left[\frac{8}{2}\arctan\left(\frac{x}{2}\right)-\frac{x^3}{12}\right]_{-2}^{2}$$

$$=\left(4\arctan 1-\frac{2}{3}\right)-\left(4\arctan(-1)+\frac{2}{3}\right)$$

$$=\left(4\frac{\pi}{4}-\frac{2}{3}\right)-\left(-4\frac{\pi}{4}+\frac{2}{3}\right)$$

$$=2\pi-\frac{4}{3}$$

Exercise 9X

1 a $v=\pi\int_0^{\frac{\pi}{2}}\cos x\,\mathrm{d}x=\pi[\sin x]_0^{\frac{\pi}{2}}=\pi\left(\sin\frac{\pi}{2}-\sin 0\right)$

$$v=\pi$$

b $v=\pi\int_0^{\frac{\pi}{2}}\sec^2 x\,\mathrm{d}x=\pi[\tan x]_0^{\frac{\pi}{4}}=\pi\left(\tan\frac{\pi}{4}-\tan 0\right)$

$$v=\pi$$

c $v=\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{b}}\cos^2 x\,2\mathrm{d}x=\frac{\pi}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{b}}(1+\cos 2x)\,\mathrm{d}x$

$$=\frac{\pi}{2}\left[x+\frac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$=\frac{\pi}{2}\left[\left(\frac{5\pi}{6}+\frac{1}{2}\sin\frac{5\pi}{3}\right)-\left(\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}\right)\right]$$

$$=\frac{\pi}{2}\left[\frac{5\pi}{6}-\frac{\sqrt{3}}{4}-\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right]=\frac{\pi}{2}\left[\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right]$$

$$=\pi\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)$$

d $v=\pi\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\sin^2 x\,\mathrm{d}x=\frac{\pi}{2}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}(1-\cos 2x)\,\mathrm{d}x$

$$=\frac{\pi}{2}\left[x-\frac{1}{2}\sin 2x\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$$

$$=\frac{\pi}{2}\left[\left(\frac{2\pi}{3}-\frac{1}{2}\sin\frac{4\pi}{3}\right)-\left(\frac{\pi}{3}-\frac{1}{2}\sin\frac{2\pi}{3}\right)\right]$$

$$=\frac{\pi}{2}\left[\frac{2\pi}{3}+\frac{\sqrt{3}}{4}-\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right]$$

$$=\frac{\pi}{2}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$$

2 **a** $v = \pi\int_0^1 \sin^2 y \, dy = \frac{\pi}{2}\int_0^1 (1 - \cos 2y)\, dy$

$$= \frac{\pi}{2}\left[y - \frac{1}{2}\sin 2y\right]_0^1$$

$$= \frac{\pi}{2}\left[\left(1 - \frac{1}{2}\sin 2\right) - 0\right]$$

$$= \frac{\pi}{2}\left(1 - \frac{1}{2}\sin 2\right)$$

b $y = \arcsin x, \quad 0 \le x \le 1$

$\Rightarrow x = \sin y, \quad 0 \le y \le \frac{\pi}{2}$

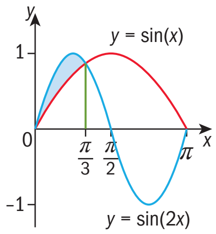
$$v = \pi\int_0^{\frac{\pi}{2}} \sin^2 y \, dy = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (1 - \cos 2y)\, dy$$

$$= \frac{\pi}{2}\left[y - \frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2} - \frac{1}{2}\sin \pi - (0)\right]$$

$$= \frac{\pi^2}{4}$$

3 **b**



$$v = \pi\int_0^{\frac{\pi}{3}} (\sin^2 2x - \sin^2 x)\, dx$$

$$\int_0^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{1}{2}\cos 4x - \frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

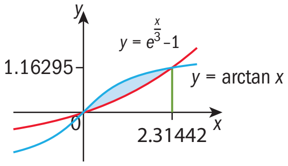
$$= \frac{\pi}{2}\int_0^{\frac{\pi}{3}} (\cos 2x - \cos 4x)\, dx$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\sin\frac{2\pi}{3} - \frac{1}{4}\sin\frac{4\pi}{3}\right]$$

$$= \frac{\pi}{2}\left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8}\right) = \frac{3\pi\sqrt{3}}{16}$$

c



$$v = \pi\int_0^{2.31442} \left((\arctan x)^2 - (e^{\frac{x}{3}} - 1)^2\right) dx = 2.35$$

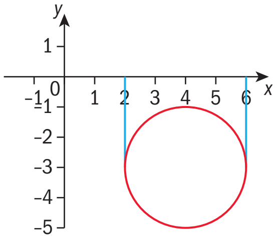
d $y = \arctan x \Rightarrow x2 + 1 \; x = \tan y$

$y = e^{\frac{x}{3}} - 1 \; e^{\frac{x}{3}} = y + 1 \; x = 3\ln(y + 1)$

$$v = \pi\int_0^{1.16295} ((3\ln (y + 1))^2 - \tan^2 y)\, dy = 4.18$$

Exercise 9Y

1



$(x - 4)^2 + (y + 3)^2 = 4$

$(y + 3)^2 = 4 - (x - 4)^2$

$$y + 3 = \pm\sqrt{4 - (x - 4)^2}$$

$$y = -3 \pm \sqrt{4 - (x - 4)^2}$$

$$V = \pi\int_2^6 \left(\left(-3 - \sqrt{4 - (x - 4)^2}\right)^2 - \left(-3 + \sqrt{4 - (x - 4)^2}\right)^2\right) dx$$

$$= \pi\int_2^6 \left(12\sqrt{4 - (x - 4)^2}\right) dx$$

$$= 12\pi\int_2^6 \sqrt{4 - (x - 4)^2}\, dx$$

Let $x - 4 = 2\sin\theta \; dx = 2\cos\theta\, d\theta$

$x = 2 \Rightarrow \sin\theta = -1 \; \theta = -\frac{\pi}{2}$

$x = 6 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$\sqrt{4 - (x - 4)^2} = 2\cos\theta$

$$v = 12\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta\, 2\cos\theta\, d\theta$$

$$= 24\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta$$

$$= 24\pi\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$v = 24\pi^2$

2 $(x - 4)^2 + (y + 3)^2 = 4$

$$(x - 4)^2 = 4 - (y + 3)^2$$

$$x - 4 = \pm\sqrt{4 - (y + 3)^2}$$

$$x = 4 \pm \sqrt{4 - (y + 3)^2}$$

$$v = \pi\int_{-5}^{-1} \left(\left(4 + \sqrt{4 - (y + 3)^2}\right)^2 - \left(4 - \sqrt{4 - (y + 3)^2}\right)^2\right) dy$$

$$= \pi\int_{-5}^{-1} 16\sqrt{4 - (y + 3)^2}\, dy$$

Let $y + 3 = 2\sin\theta \quad dy = 2\cos\theta\, d\theta$

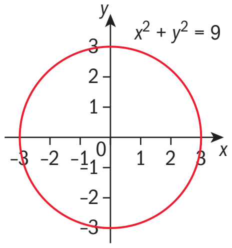
$y = -5 \Rightarrow \sin\theta = -1 \Rightarrow \theta = -\frac{\pi}{2}$

$y = -1 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$\sqrt{4 - (y + 3)^2} = 2\cos\theta$

$v = 16\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta\, 2\cos\theta\, d\theta = 32\pi^2$(see qn.1)

3



$y^2 = 9 - x^2$

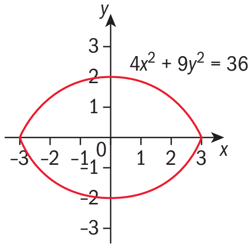
$$v = \pi\int_{-3}^{3}(9 - x^2)\,\mathrm{d}x$$

$$= \pi\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$

$$v = \pi\left[\left(27 - \frac{27}{3}\right) - \left(-27 + \frac{27}{3}\right)\right]$$

$v = 36\pi$

4



$4x^2 + 9y = 36$

$$y^2 = 4 - \frac{4}{9}x^2$$

$$v = \pi\int_{-3}^{-3}\left(4 - \frac{4}{9}x^2\right)\mathrm{d}x$$

$$v = \pi\left[4x - \frac{4x^3}{27}\right]_{-3}^{3}$$

$= \pi\,[(12 - 4) - (-12 + 4)] = 16\pi$

5 $x^2 = 9 - \frac{9}{4}y^2$, $v = \pi\int_{-2}^{2}\left(9 - \frac{9}{4}y^2\right)\mathrm{d}y$

$$v = \pi\left[9y - \frac{3}{4}y^3\right]_{-2}^{2}$$

$\pi[(18 - 6) - (-18 + 6)] = 24\pi$

Review exercise

1 **a** $f(x) = (2x+3)\sin x$

$\Rightarrow f'(x) = 2\sin x + (2x+3)\cos x$

b $g(x) = \mathrm{e}^x\cos 3x$

$\Rightarrow g'(x) = \mathrm{e}^x\cos 3x + \mathrm{e}^x\cdot(-\sin 3x)\cdot 3$

$= \mathrm{e}^x(\cos 3x - 3\sin 3x)$

c $h(x) = \frac{\tan x}{2x^2} = \frac{1}{2}\tan x\cdot x^{-2}$

$$\Rightarrow h'(x) = \frac{1}{2}\sec^2 x\cdot x^{-2} + \frac{1}{2}\tan x\cdot\left(-2x^{-3}\right)$$

$$= \frac{x - 2\sin x\cos x}{2x^3\cos^2 x}$$

$$= \frac{x - \sin 2x}{2x^3\cos^2 x}$$

2 $\sin y + \mathrm{e}^{2x} = 1 \Rightarrow \cos y\cdot y' + \mathrm{e}^{2x}\cdot 2 = 0.$

$$\Rightarrow y' = \frac{-2\mathrm{e}^{2x}}{\cos y}$$

$$\Rightarrow m = y'(0) = \frac{-2\mathrm{e}^0}{\cos 0} = -2$$

$T: y - 0 = -2(x - 0) \Rightarrow y = -2x$

3 $$\int_{\frac{\pi}{4}}^{m}\sec^2 x\,\mathrm{d}x = [\tan x]_{\frac{\pi}{4}}^{m} = \tan m - \tan\frac{\pi}{4}$$

$$= \tan m - 1 = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$$

$$\Rightarrow \tan m - 1 = \sqrt{3} - 1 \Rightarrow \tan m = \sqrt{3} \Rightarrow m = \frac{\pi}{3}$$

4 **a** $\int(2x-5)\mathrm{e}^{2x}\mathrm{d}x =$ Let $2x - 5 = u$, $\mathrm{e}^{2x}\mathrm{d}x = \mathrm{d}v$ $\Rightarrow$ $2\mathrm{d}x = \mathrm{d}u$, $\frac{1}{2}\mathrm{e}^{2x} = v$,

$$\frac{2x-5}{2}\mathrm{e}^{2x} - \int\mathrm{e}^{2x}\mathrm{d}x = \left(x - \frac{5}{2}\right)\mathrm{e}^{2x} - \frac{1}{2}\mathrm{e}^{2x} + c$$

$$= (x - 3)\mathrm{e}^{2x} + c$$

b

	dv = cosx dx	**sign**
***u* = *x*² – 5**	*v* = sinx	+
2*x*	−cosx	–
2	−sinx	+

$(x^2 - 5x)\cos x\,\mathrm{d}x = (x^2 - 5x)\sin x$

$+ 2x\cos x - 2\sin x + c;$

$= (x^2 - 5x - 2)\sin x + 2x\cos x + c$

c

	dv = eˣdx	**sign**
***u* = cosx**	*v* = eˣ	+
−3sin3x	eˣ	–
−9cos3x	eˣ	+

$\int\mathrm{e}^x\cos 3x\,\mathrm{d}x = \mathrm{e}^x\cos 3x + \mathrm{e}^x 3\sin 3x$

$- 9\int\mathrm{e}^x\cos 3x\,\mathrm{d}x.$

$10\int\mathrm{e}^x\cos 3x\,\mathrm{d}x = \mathrm{e}^x\cos 3x + 3\mathrm{e}^x\sin 3x$

$$\Rightarrow \int\mathrm{e}^x\cos 3x\,\mathrm{d}x = \frac{\mathrm{e}^x}{10}(\cos 3x + 3\sin 3x)$$

5 $A = \frac{1}{2}d^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = d\cdot\frac{\mathrm{d}d}{\mathrm{d}t}$

$\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = \sqrt{5}\cdot 0.2 = \frac{\sqrt{5}}{5}$ cm²/s.

6 The curve $y = \mathrm{e}^{2x-1}$ is given.

a $y = \mathrm{e}^{2x-1} \Rightarrow y' = \mathrm{e}^{2x-1}\cdot 2$

$m = y'(x_0) = \mathrm{e}^{2x_0-1}\cdot 2 \Rightarrow T: y - y_0 = m(x - x_0)$

$y = 2\mathrm{e}^{2x_0-1}(x - x_0) + y_0 \Rightarrow$

$$y = 2\mathrm{e}^{2x_0-1}x\underbrace{-2\mathrm{e}^{2x_0-1}x_0 + \mathrm{e}^{2x_0-1}}_{0}$$

$\mathrm{e}^{2x_0-1}(-2x_0 + 1) = 0 \Rightarrow x_0 = \frac{1}{2}$

$T: y = 2x$

b $\int_0^{\frac{1}{2}} \left(e^{2x-1} - 2x\right)dx = \left[\frac{1}{2}e^{2x-1} - x^2\right]_0^{\frac{1}{2}}$

$= \frac{1}{2} - \frac{1}{4} - \frac{1}{2e} = \frac{e-2}{4e}$

c $\int_0^{\frac{1}{2}} \left(\left(e^{2x-1}\right)^2 - (2x)^2\right)dx = \pi\int_0^{\frac{1}{2}} \left(e^{4x-2} - 4x^2\right)dx$

$= \pi\left[\frac{1}{4}e^{4x-2} - \frac{4}{3}x^3\right]_0^{\frac{1}{2}}$

$= \pi\left(\frac{1}{4} - \frac{1}{6} - \frac{1}{4e^2}\right) = \frac{\left(e^2-3\right)\pi}{12e^2}$

7 Let $x = 3\cos\theta \Rightarrow dx = -3\sin\theta d\theta, \quad \theta = \arccos\left(\frac{x}{3}\right)$

$\sqrt{9 - x^2} = \sqrt{9 - 9\cos^2\theta} = 3\sin\theta$

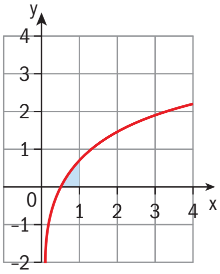
$\int\sqrt{9-x^2}\,dx = -9\int \sin^2\theta d\theta - 9\int \frac{1-\cos 2\theta}{2}d\theta$

$= -9\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]$

$= \frac{9}{2}\sin\left(\arccos\left(\frac{x}{3}\right)\right)\cos\left(\arccos\left(\frac{x}{3}\right)\right) - \frac{9}{2}\arccos\left(\frac{x}{3}\right) + c$

$= \frac{x}{2}\sqrt{9-x^2} - \frac{9}{2}\arccos\left(\frac{x}{3}\right) + c$

8 **a**



b $y = \ln(2x) \Rightarrow x = \frac{1}{2}e^y, \quad x = 1 \Rightarrow y = \ln 2$

$\pi\int_0^{\ln 2}\left(1 - \left(\frac{1}{2}e^y\right)^2\right)dy = \pi\int_0^{\ln 2}\left(1 - \frac{e^{2y}}{4}\right)dy$

$= \pi\left[y - \frac{1}{8}e^{2y}\right]_0^{\ln 2}$

$= \pi\left(\ln 2 - \frac{4-1}{8}\right) = \frac{(8\ln 2 - 3)\pi}{8}$

9 **a** $s = \int_0^k v\,dt = \int_0^k 5e^{-\frac{2t}{3}}dt = 5\left[-\frac{3}{2}e^{-\frac{2t}{3}}\right]_0^k$

$= \frac{15}{2}\left(1 - e^{-\frac{2k}{3}}\right)$

b $\lim_{k\to\infty} s = \lim_{k\to\infty}\left(\frac{15}{2} - \frac{15}{2}e^{-\frac{2k}{3}}\right) = \frac{15}{2} - 0 = 7.5\,\text{m}$

10 $x^2y^3 = \cos(\pi x) \Rightarrow 2xy^3 + x^2 3y^2 y' = -\sin(\pi x)\cdot\pi$

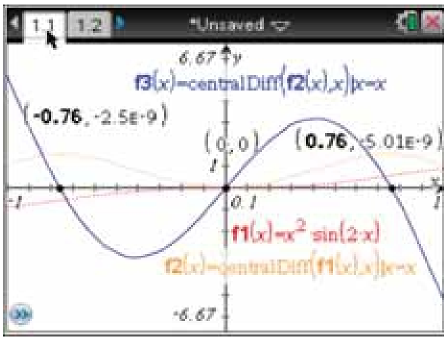
$2\cdot 1(-1)^3 + 1^2\cdot 3(-1)^2\cdot m_T = -\sin(\pi)\cdot\pi$

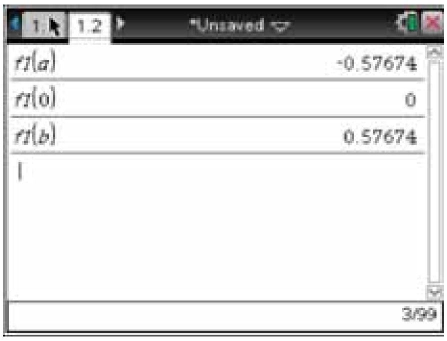
$\Rightarrow m_T = \frac{2}{3} \Rightarrow m_N = -\frac{3}{2}$

$N: y + 1 = -\frac{3}{2}(x-1) \Rightarrow y = -\frac{3}{2}x + \frac{1}{2}$

Review exercise

1 We need to find the zeros of the second derivative of the function $y = x^2\sin 2x, -1 \le x \le 1$. We store the variables a and b and then to find the y-coordinates of the points of inflexion we input those values of x in the original function.





f1(a)	-0.57674
f1(0)	0
f1(b)	0.57674

So the points of inflexion are $(-0.760, -0.577)$, $(0, 0)$ and $(0.760, 0.577)$.

2 $y^3 = \cos x \Rightarrow 3y^2y' = -\sin x \Rightarrow y' = -\frac{\sin x}{3y^2}$

When $x = 1 \Rightarrow y^3 = \cos 1 \Rightarrow y = \sqrt[3]{\cos 1} = 0.814$

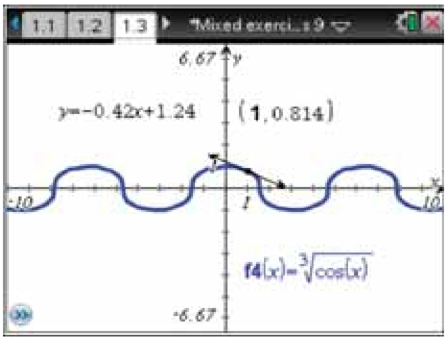
$m = -\frac{\sin 1}{3\sqrt[3]{\cos^2 1}} = -0.423$

$T: y - 0.814 = -0.423(x - 1) \Rightarrow$

$y = -0.423x + 1.24$

Checking:

First we find the explicit form of the curve and graph it on a GDC. $y^3 = \cos x \Rightarrow y = \sqrt[3]{\cos x}$



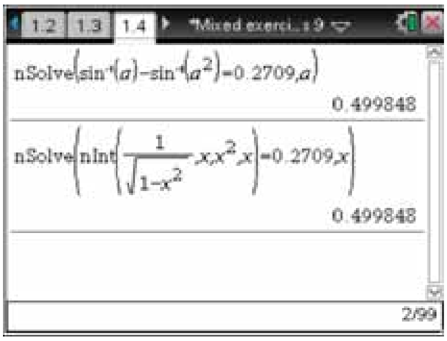
3 Find the value of a, $0 < a < 1$, such that

$$\int_{a^2}^{a} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = 0.2709$$

$$\int_{a^2}^{a} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \left[\arcsin x\right]_{a^2}^{a} = \arcsin a - \arcsin a^2$$
$$= 0.2709$$

We use a GDC to solve this equation, $a = 0.500$.

The result can be obtained directly on a calculator by using numerical integration.

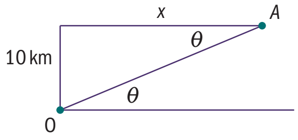


$$\text{nSolve}\left(\sin^{-1}(a)-\sin^{-1}(a^2)=0.2709,a\right)$$
0.499848

$$\text{nSolve}\left(\text{nInt}\left(\frac{1}{\sqrt{1-x^2}},x,x^2,x\right)=0.2709,x\right)$$
0.499848

2/99

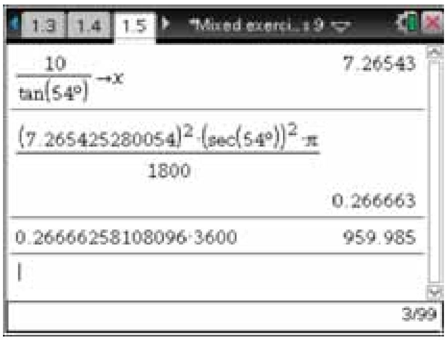
4



$$\tan\theta = \frac{10}{x} \Rightarrow x = \frac{10}{\tan\theta} = \frac{10}{\tan 54°} = 7.27$$

$$\tan\theta = \frac{10}{x} \Rightarrow \sec^2\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{10}{x^2}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x^2\sec^2\theta}{10}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{7.27^2\cdot\sec^2 54°}{10}\cdot\frac{\pi}{180} = -0.267\ \text{km}^{-1}$$



$$\frac{10}{\tan(54°)} \to x$$
7.26543

$$\frac{(7.265425280054)^2\cdot(\sec(54°))^2\cdot\pi}{1800}$$
0.266663

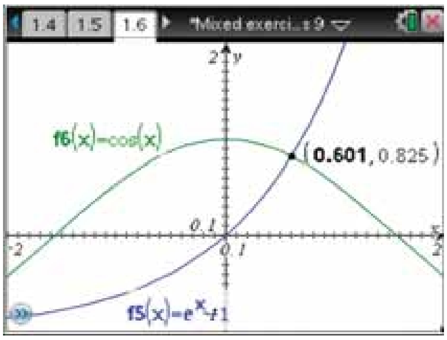
0.26666258108096·3600 959.985

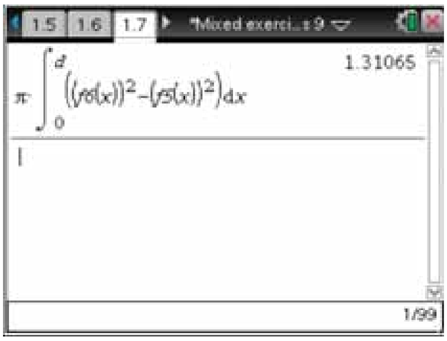
3/99

So the speed of the plane is 960 km/h.

5 First we graph the functions and identify the region. Then we find the point of intersection between the curves and store the x-coordinate to a variable called d.

$$V = \pi\int_{0}^{0.601}\left(\cos^2 x - \left(e^x - 1\right)^2\right)\mathrm{d}x = 1.31$$





$$\pi\cdot\int_{0}^{d}\left((f6(x))^2-(f5(x))^2\right)\mathrm{d}x$$
1.31065

1/99

10 Modeling randomness

Skills check

1 **a** $\frac{1}{8}$ **b** $\frac{5}{36}$

2 **a** 0.8 **b** 0.2 + 0.3 – 0.1 = 0.4

Exercise 10A

1 **a** No, because $\sum f(x) = 1.1$, not 1

b No, as above

c No, because $f(-1)$ is negative.

2 **a** $a = 1 - 0.88 = 0.12$

b $P(1 \le X \le 3) = P(1) + P(2) + P(3) = 0.87$

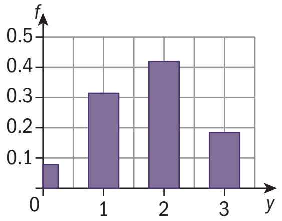
c $P(X \le 3) = 0.87$

3 **a** $\sum_{0}^{3} f(t) = 1 \Rightarrow 4k + 3k + 2k + k = 1 \Rightarrow k = \frac{1}{10}$

b $P(1 \le T < 3) = P(1) + P(2) = 3k + 2k = \frac{1}{2}$

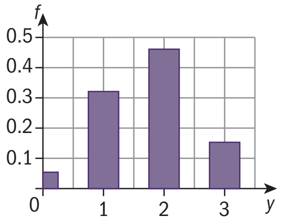
4

x	0	1	2	3
P(*X* = *x*)	$\frac{27}{343}$	$\frac{108}{343}$	$\frac{144}{343}$	$\frac{64}{343}$



5

x	0	1	2	3
P(*X* = *x*)	$\frac{120}{2184}$	$\frac{702}{2184}$	$\frac{1008}{2184}$	$\frac{336}{2184}$



Exercise 10B

1 **a** $E(R) = 1 \times \frac{1}{5} + 5 \times \frac{2}{5} + 10 \times \frac{2}{5} = \frac{31}{5} = 6.2$

b $E(R^2) = 1 \times \frac{1}{5} + 25 \times \frac{2}{5} + 100 \times \frac{2}{5} = 50.2$

c $\mathrm{Var}(R) = E(R^2) - [E(R)]^2 = 50.2 - 6.2^2 = 11.76$

d Standard deviation $= \sqrt{11.76} = 3.43\,(3\,\mathrm{sf})$

2 **a** $P(X \ge 2) = 3P(X < 2)$

$\Rightarrow \quad a + 3b = 3 \times 2a \quad \Rightarrow \quad 3b = 5a$

But $\sum P(X) = 1$ so $3a + 3b = 1$

$\therefore \quad 1 - 3a = 5a$ so $a = \frac{1}{8}$ and $b = \frac{5}{24}$

b
$$E(X) = 0 \times \frac{1}{8} + 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + 3 \times \frac{5}{24} + 4 \times \frac{5}{24} + 5 \times \frac{5}{24}$$
$$= \frac{23}{8}$$
$$E(X^2) = 0 \times \frac{1}{8} + 1 \times \frac{1}{8} + 4 \times \frac{1}{8} + 9 \times \frac{5}{24} + 16 \times \frac{5}{24} + 25 \times \frac{5}{24}$$
$$= \frac{265}{24}$$

c
$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$
$$= \frac{265}{24} - \left(\frac{23}{8}\right)^2$$
$$= \frac{533}{192}$$

3 **a** P(bottom > top) = P(top > bottom).

Since bottom cannot equal top,

P(bottom > top) $= \frac{1}{2}$

b $P(S = 4) = P(1, 3) + P(3, 1)$
$$= \frac{1}{10} \times \frac{1}{9} + \frac{1}{10} \times \frac{1}{9} = \frac{1}{45}$$

c

x	**P(S = *x*)**
3	$\frac{1}{45}$
4	$\frac{1}{45}$
5	$\frac{2}{45}$
6	$\frac{2}{45}$
7	$\frac{3}{45}$
8	$\frac{3}{45}$
9	$\frac{4}{45}$
10	$\frac{4}{45}$
11	$\frac{1}{9}$
12	$\frac{4}{45}$
13	$\frac{4}{45}$
14	$\frac{3}{45}$
15	$\frac{3}{45}$
16	$\frac{2}{45}$
17	$\frac{2}{45}$
18	$\frac{1}{45}$
19	$\frac{1}{45}$

d $E(S)=3\times\frac{1}{45}+4\times\frac{1}{45}+\ldots+18\times\frac{1}{45}+19\times\frac{1}{45}$

$=\frac{495}{45}=11$ (also obvious from symmetry)

$\text{Var}(S)=\frac{1}{45}(9+16+25\times2+36\times2+49$
$\times3+64\times3+81\times4+100\times4+121$
$\times5+144\times4+169\times4+196\times3+225$
$\times3+256\times2+289\times2+324+361)-11^2$

$=\frac{6105}{45}=14\frac{2}{3}$

4 Let £A be the amount paid.
Then if W = your winnings,

$E(W)=20\times0.2+10\times0.4+(-A)\times0.4$

$E(W)=0,\quad$ so $\quad 4+4-0.4A=0$

$\Rightarrow 0.4A=8 \quad \Rightarrow A=20$

So you pay £20

5 **a** $\sum_{c=1}^{7} f(t)=1 \Rightarrow k+4k+9k+16k+9k+4k+k=1$

$\Rightarrow 44k=1 \Rightarrow k=\frac{1}{44}$

b $P(T=4)=k(8-4)^2=16k=\frac{16}{44}=\frac{4}{11}$

$P(T\le4)=k+4k+9k+16k=\frac{30}{44}=\frac{15}{22}$

$P(T=4|T\le4)=\frac{P(T=4\text{ and }T\le4)}{P(T\le4)}=\frac{P(T=4)}{P(T\le4)}$

$=\frac{\frac{4}{11}}{\frac{15}{22}}=\frac{8}{15}$

c $E(T)=\sum_{1}^{7} tf(t)$

$=1\times k+2\times4k+3\times9k+4\times16k+5$
$\times9k+6\times4k+7\times k$

$=\frac{176}{44}=4$

$\text{Var}(t)=E(T^2)-4^2$

$=1\times k+4\times4k+9\times9k+16\times16k+25\times9k$
$+36\times4k+49\times k-16$

$=\frac{772}{44}-16=\frac{17}{11}$

d Mode of $T=4$ (highest probability)

Exercise 10C

1 **a** $P(X\le15)=\frac{16}{15}=\frac{2}{5}$

b

x	5	10	15	20	25	30
***f*(*x*) = P(*X* ≤ *x*)**	$\frac{1}{15}$	$\frac{3}{15}$	$\frac{6}{15}$	$\frac{10}{15}$	$\frac{13}{15}$	1

c From table, median = 20

2 **a** $P(\text{at least 3 lines})=P(L\ge3)$

$=0.31+0.12+0.04=0.47$

b $E(L)=0\times0.07+1\times0.21+2\times0.25+3\times0.31$
$+4\times0.12+5\times0.04$
$=2.32$

$\text{Var}(L)=E(L^2)-2.32^2$

$=6.92-5.3824=1.5376$

c

x	0	1	2	3	4	5
***f*(*x*) = P(*X*≤*x*)**	0.07	0.28	0.53	0.84	0.96	1

d From table, median = 2

3 **a**

x	2	3	4	5	6
***f*(*x*) = P(*X* = *x*)**	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$
***f*(*x*) = P(*X* ≤ *x*)**	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	1

b Mean $=E(S)=2\times\frac{1}{36}+3\times\frac{4}{36}+4\times\frac{10}{36}+5\times\frac{12}{36}+6\times\frac{9}{36}$

$=\frac{2+12+40+60+54}{36}$

$=\frac{168}{36}=4\frac{2}{3}$

From table, median = 5, mode = 5

c $\text{Var}(X)=E(X^2)-\left(4\frac{2}{3}\right)^2$

$=4\times\frac{1}{36}+9\times\frac{4}{36}+16\times\frac{10}{36}+25\times\frac{12}{36}+36\times\frac{9}{36}$

$=22\frac{8}{9}-21\frac{7}{9}=1\frac{1}{9}$

Standard deviation $=\sqrt{1\frac{1}{9}}=1.05$ (3 sf)

4 **a** $\sum_{k=1}^{n} kx=1 \quad\Rightarrow\quad k(1+2+\ldots+n)=1$

$\Rightarrow\quad k\frac{n}{2}(n+1)=1 \quad\Rightarrow\quad k=\frac{2}{n^2+n}$

b $E(X)=1\times k+2\times2k+3\times3k+\ldots+n\times nk$

$=k(1^2+2^2+3^2+\ldots+n^2)$

$=k\frac{n}{6}(n+1)(2n+1)$

$=\frac{2}{n(n+1)}\times\frac{n}{6}(n+1)(2n+1)$

$=\frac{2n+1}{3}$

5 **a** $\sum_{1}^{\infty} f(x)=1 \quad\Rightarrow 3^{a-1}+3^{a-2}+3^{a-3}+\ldots=1$

$\Rightarrow 3^{a-1}=\left(1+\frac{1}{3}+\frac{1}{9}+\ldots\right)=1$

$\Rightarrow 3^{a-1}\times\frac{1}{1-\frac{1}{3}}=1 \quad\Rightarrow\quad 3^{a-1}\times\frac{3}{2}=1$

$\Rightarrow 3^a=2 \quad\Rightarrow a=\log_3 2$

b $\mathrm{F}(x) = \mathrm{P}(X \le x) = \sum_{r=1}^{x} 3^{a-r}$

$= 3^{a-1} + 3^{a-2} + \ldots + 3^{a-x}$

$= 3^{a-1}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots + \frac{1}{3^{x-1}}\right)$

$= 3^{a-1} \times \dfrac{1\left(1-\left(\frac{1}{3}\right)^x\right)}{1-\frac{1}{3}}$

$= 3^{a-1} \times \frac{3}{2}\left(1 - \frac{1}{3^x}\right) = 1 - \frac{1}{3^x}$

$\therefore \mathrm{F}(x) = 1 - 3^{-x},\ x = 1, 2, 3$

Exercise 10D

1 $X \sim \mathrm{B}\left(10, \frac{1}{2}\right)$

a $10C4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^6 = \frac{210}{1024} = \frac{105}{512} \approx 0.205$

b $\mathrm{P}(X \ge 6) = 1 - \mathrm{P}(X \le 5) = 1 - 0.6230 \approx 0.377$

c $\mathrm{P}(X \le 5) \approx 0.623$

2 **a** $X \sim \mathrm{B}(5, 0.6)$

$\mathrm{P}(X = 2) = 5C2\,(0.6)^2(0.4)^3 \approx 0.230$

b $X \sim \mathrm{B}(7, 0.6)$

$\mathrm{P}(X \ge 3) = 1 - \mathrm{P}(X \le 3) \approx 0.904$

c $X \sim \mathrm{B}(9, 0.6)$

$\mathrm{P}(X \ge 5) \approx 0.733$

3 $X \sim \mathrm{B}(8, 0.01)$

a **i** $\mathrm{P}(X \ge 1) = 1 - \mathrm{P}(X = 0) = 1 - (0.99)^8 = 0.077$

ii $\mathrm{P}(X \le 2) = \mathrm{P}(0) + \mathrm{P}(1) + \mathrm{P}(2)$

$= (0.99)^8 + 8 \times (0.99)^7 \times (0.01) + 28 \times (0.99)^6 \times (0.01)^2$

$\approx 0.9999 \quad (4\text{ sf})$

b $\mathrm{P}(X = 2 \mid X \ge 1) = \dfrac{\mathrm{P}(X = 2 \text{ and } X \ge 1)}{\mathrm{P}(X \ge 1)}$

$= \dfrac{\mathrm{P}(X = 2)}{\mathrm{P}(X \ge 2)} = \dfrac{28 \times 0.99^6 \times 0.01^2}{0.077} \approx 0.034$

4 B(6, 0.35)

x	0	1	2	3	4	5	6
P(*x*)	0.0754	0.2437	0.328	0.2355	0.0951	0.0205	0.0018

a Mode *a* = 2

b Median *b* = 2

c $\mathrm{P}(X < 4 \mid X > 2) = \dfrac{\mathrm{P}(X < 4 \text{ and } X > 2)}{\mathrm{P}(X > 2)}$

$= \dfrac{\mathrm{P}(X = 3)}{1 - \mathrm{P}(0) - \mathrm{P}(1) - \mathrm{P}(2)} = \dfrac{0.2355}{0.3529} \approx 0.667$

5 **a**

***P* (*X*≤*x*)**	***x* = 0**	***x* = 1**	***x* = 2**	***x* = 3**	***x* = 4**	***x* = 5**	***x* = 6**	***x* = 7**	***x* = 8**	***x* = 9**	***x* = 10**
n = 2	0.36	0.84	1								
n = 5	0.07776	0.33696	0.68256	0.91296	0.98976	1					
n = 10	0.00605 (3 sf)	0.0464 (3 sf)	0.167 (3 sf)	0.382 (3 sf)	0.633 (3 sf)	0.834 (3 sf)	0.945 (3 sf)	0.98771 (3 sf)	0.99832 (3 sf)	0.99990 (3 sf)	1 (3 sf)

b 26

6 **a** $X \sim \mathrm{B}(n, 0.3)$

$\mathrm{P}(X > 3) > 0.7 \quad \Rightarrow \quad \mathrm{P}(X \le 3) < 0.3 \quad \Rightarrow \quad n = 15$

7 **a** $(0.45)^7 \approx 0.00374$

b $X \sim \mathrm{B}(7, 0.45)$

$\mathrm{P}(X \ge 2) = 1 - \mathrm{P}(X \le 1)$

$= 1 - 0.1024 \approx 0.8976$

c $5 \times (0.45)^3 \times (0.55)^4 \approx 0.0417$, as there are only 5 sequences where it rains on 3 consecutive days

8 $X \sim \mathrm{B}(n, 0.6)$

Need $\mathrm{P}(X \ge 1) > 0.95 \quad \Rightarrow \quad \mathrm{P}(X \le 0) < 0.05$

$\Rightarrow \quad (0.4)^n < 0.05 \quad \Rightarrow \quad n = 4$

Exercise 10E

1 $X \sim \mathrm{B}(8, 0.4)$

a $\mathrm{P}(X = 5) = 8C5(0.4)^5(0.6)^3 \approx 0.124$

b $\mathrm{P}(X \le 5) \approx 0.950$

c $\mathrm{P}(X < 5) \approx 0.826$

d $\text{Mean} = \mathrm{E}(X) = np = 8 \times 0.4 = 3.2$

e $\text{Variance} = npq = 8 \times 0.4 \times 0.6 = 1.92$

2 **a** $\mathrm{P}(Y = 1) + \mathrm{P}(Y = 2) = \mathrm{P}(Y \le 2) - \mathrm{P}(Y \le 0)$

$= 0.6471 - 0.0824$

≈ 0.565

b $\mathrm{P}(Y \le 2) \approx 0.647$

c $\mathrm{P}(Y \ge 2) = 1 - \mathrm{P}(Y \le 1) = 1 - 0.3294 \approx 0.671$

d Median = 2 (first value of x for which CDF ≥ 0.5)

3 **a**

x	0	1	2	3	4	5
f(*x*) = P (*X*=*x*)	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{1}{32}$
F(*x*) = P (*X*≤ *x*)	$\frac{1}{32}$	$\frac{3}{16}$	$\frac{1}{2}$	$\frac{13}{16}$	$\frac{31}{32}$	1

b Mode of T = 2 or 3 (bimodal)

c $\text{Median} = \frac{2+3}{2} = 2.5$

4 **a** $30C3(0.02)^3(0.98)^{27} \approx 0.0188$

b $(0.98)^5 \approx 0.9039$

c $\text{Mean} = \mathrm{E}(X) = np = 30 \times 0.02 = 0.6$

5 $np = 2 \quad (1)$

$np = 1.5 \quad (2)$

$(2) \div (1) \Rightarrow \quad 1 - p = \frac{1.5}{2} = \frac{3}{4}$

$\Rightarrow p = \frac{1}{4}$ and $n = 8$

6 **a** **i** $\left(\frac{3}{4}\right)^{20} \approx 0.0032$

ii $X \sim \mathrm{B}\left(20, \frac{1}{4}\right)$

$\mathrm{P}(X > 10) = 1 - \mathrm{P}(X \le 10) = 1 - 0.9961 \approx 0.0039$

iii $\mathrm{P}(X \le 5) \approx 0.6172$

b $\mathrm{E}(X) = np = 20 \times \frac{1}{4} = 5$

Standard deviation $= \sqrt{\mathrm{Var}(X)} = \sqrt{npq}$

$= \sqrt{20 \times \frac{1}{4} \times \frac{3}{4}} = \frac{1}{4}\sqrt{60} \approx 1.94$

c $Y \sim \mathrm{B}(5, 0.0039)$

$\mathrm{P}(Y \ge 2) = 1 - \mathrm{P}(0) - \mathrm{P}(1) = 1 - (0.9961)^5$

$\quad - 5 \times (0.9961)^4 \times 0.0039 \approx 0.0002$

7 $X \sim \mathrm{B}(10, 0.18)$

a $\mathrm{P}(X = 2) = 10C2\,(0.18)^2\,(0.82)^8 \approx 0.298$

b $\mathrm{P}(X \ge 1) = 1 - P(0) = 1 - (0.82)^{10} \approx 0.863$

c Most likely value = mode = 1

d $Y \sim \mathrm{B}(25, 0.18)$

$\therefore \mathrm{E}(Y) = np = 25 \times 0.18 = 4.5$

e $\mathrm{Var}(Y) = npq = 25 \times 0.18 \times 0.82 = 3.69$

f Require $\mathrm{P}(Z \ge 2) > 0.95$ where $Z \sim \mathrm{B}(n, 0.18)$

i.e. $1 - \mathrm{P}(0) - \mathrm{P}(1) > 0.95$

$\Rightarrow 1 - (0.82)^n - n \times (0.82)^{n-1} \times 0.18 > 0.95$

$\Rightarrow (0.82)^n + n \times (0.82)^{n-1} \times 0.18 < 0.05$

$\Rightarrow (0.82)^{n-1}(0.82 + 0.18n) < 0.05$

$\Rightarrow n = 25$

g Men: $M \sim \mathrm{B}(5, 0.22)$

Women: $W \sim \mathrm{B}(5, 0.16)$

$$\mathrm{P}(W \ge 1) \times \mathrm{P}(M \ge 1) = [1 - \mathrm{P}(W = 0)] \times [1 - \mathrm{P}(M = 0)]$$
$$= \left[1 - (0.84)^5\right] \times \left[1 - (0.78)^5\right]$$
$$= 0.414$$

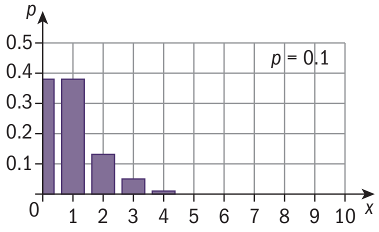
8 **a** $X \sim \mathrm{B}(5, 0.3)$

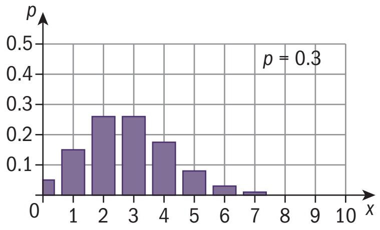
i $\mathrm{P}(X = 4) = 5C4 \times (0.3)^4 \times 0.7 \approx 0.0567$

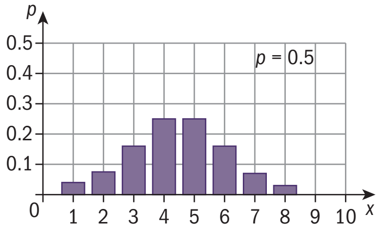
ii $\mathrm{P}(X \ge 3) = 1 - \mathrm{P}(X \le 2) = 1 - 0.0369$

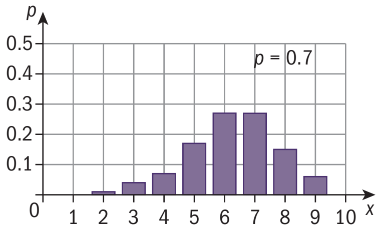
$= 0.1631$

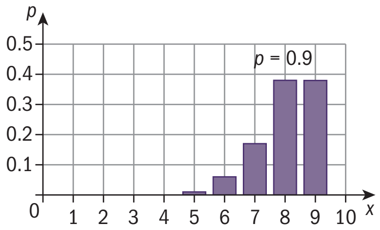
9 **a**











b The graph is symmetrical when $p = 0.5$ and asymmetrical with respect to the line $x = 4.5$ otherwise. For $p < 0.5$ the graph is positively skewed and for $p > 0.5$ it is negatively skewed. The graphs for values of p that add up to 1 are reflections of each other in the line $x = 4.5$

c

	mean	mode	median
$p = 0.1$	0.9	0 and 1	1
$p = 0.3$	2.7	2 and 3	3
$p = 0.5$	4.5	4 and 5	4.5
$p = 0.7$	6.3	6 and 7	6
$p = 0.9$	8.1	8 and 9	8

The values of the parameters of the distributions reflect the symmetries observed (eg. the sum of the means for $p = 0.1$ and $p = 0.9$ is 9).

10 **a**

w	0	1	2	4
$P(W = w)$	$(1-a)^2(1-2b)$	4ab $(1-a)(1-b)$	2ab $(a+b-2ab)$	a^2b^2

b $E(W) = \sum x\, P(W = x) = 4ab\,(1 - a)(1 - b)$

$\quad + 4ab\,(a + b - 2ab)$

$\quad + 4a^2b^2$

$= 4ab(1 - a - b + ab + a + b - 2ab) + 4a^2b^2$

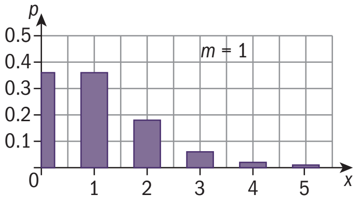
$= 4ab(1 - ab + ab) = 4ab$

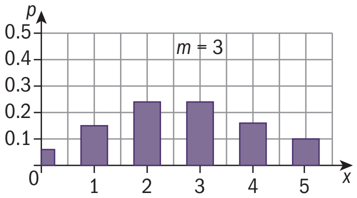
Exercise 10F

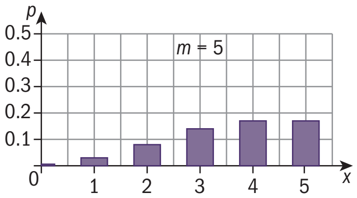
1

P(*X* = *x*)	0	1	2	3	4	5
***m* = 1**	0.368	0.368	0.184	0.0613	0.0153	0.00307
***m* = 3**	0.0498	0.149	0.224	0.224	0.168	0.101
***m* = 5**	0.0067	0.0337	0.0842	0.140	0.176	0.176

(to 3 sf)







2 **a** $\mathrm{P}(Y=3)=\frac{\mathrm{e}^{-3}3^3}{3!}\approx 0.224$

b $\mathrm{P}(Y<3)=\mathrm{P}(0)+\mathrm{P}(1)+\mathrm{P}(2)\approx 0.423$

c $\mathrm{P}(Y>3)=1-\mathrm{P}(Y\le 3)=1-0.6472\approx 0.353$

d $\mathrm{P}\left(Y=4\middle|Y>3\right)=\frac{\mathrm{P}(Y=4\,\text{and}\,Y>3)}{\mathrm{P}(Y>3)}$

$=\frac{\mathrm{P}(Y=4)}{\mathrm{P}(Y>3)}\approx 0.476$

Exercise 10G

1 **a** $\mathrm{P}(X=2)=\frac{\mathrm{e}^{-m}m^2}{2!}=\frac{\mathrm{e}^{-0.7}\times 0.49}{2}\approx 0.122$

b $\mathrm{P}(X\ge 2)=1-\mathrm{P}(0)-\mathrm{P}(1)=1-\mathrm{e}^{-0.7}\times 0.7$

$=1-1.7\mathrm{e}^{-0.7}\approx 0.156$

2 **a** $\mathrm{P}(X\ge 4)=1-\mathrm{P}(X\le 3)=1-0.6472\approx 0.353$

b $\mathrm{P}(\mathrm{P}\le 2)\approx 0.423$

Exercise 10H

1 **a** $X\sim \mathrm{P}_0(7),\ \mathrm{P}(X\le 6)=0.4497$

b $Y\sim \mathrm{P}_0(1.75),\mathrm{P}(Y\ge 2)=1-\mathrm{P}(0)-\mathrm{P}(1)$

$=1-\mathrm{e}^{-1.75}-\mathrm{e}^{-1.75}\times 1.75$

$=1-2.75\mathrm{e}^{-1.75}$

≈ 0.5221

2 **a** $X\sim \mathrm{P}_0(1)$

$\mathrm{P}(X=2)=\frac{e^{-m}m^2}{2!}=\frac{\mathrm{e}^{-1}}{2}\approx 0.184$

b $Y\sim \mathrm{P}_0(0.25)$

$\mathrm{P}(Y\ge 1)=1-\mathrm{P}(0)=1-\mathrm{e}^{-0.25}\approx 0.221$

3 **a** $X\sim \mathrm{P}_0(3)\Rightarrow \mathrm{E}(X)=3$

b $Y\sim \mathrm{P}_0(2)\Rightarrow \mathrm{P}(Y>5)=1-\mathrm{P}(Y\le 5)=1-0.9834$

≈ 0.017

4 **a** **i** $\mathrm{P}(X=3)=\frac{\mathrm{e}^{-3.5}(3.5)^3}{3!}\approx 0.216$

ii $\mathrm{P}(X>3)=1-\mathrm{P}(X\le 3)=1-0.5366\approx 0.463$

iii $\mathrm{P}\left(X<5\middle|X>3\right)=\frac{\mathrm{P}(X<5\,\text{and}\,X>3)}{\mathrm{P}(X>3)}$

$=\frac{\mathrm{P}(X=4)}{\mathrm{P}(X>3)}=\frac{0.1888}{0.4634}\approx 0.407$

b $\mathrm{E}(X)=m=3.5,\ \mathrm{Var}(X)=m=3.5$

c $\mathrm{E}(X^2)-(\mathrm{E}(X))^2=\mathrm{Var}(X)$

$\Rightarrow \mathrm{E}(X^2)-(3.5)^2=3.5$

$\Rightarrow\quad \mathrm{E}(X^2)=15.75$

5 **a** $\mathrm{P}(0)+\mathrm{P}(1)-\mathrm{P}(4)=0$

$\Rightarrow \mathrm{e}^{-m}+\mathrm{e}^{-m}m-\frac{\mathrm{e}^{-m}m^4}{4!}=0$

$\Rightarrow 1+m-\frac{m^4}{24}=0$

$\Rightarrow m^4=24(1+m)$

$\Rightarrow m=3.2\ \ (2\text{ sf})$

6 $\mathrm{P}(X>3)=0.555$

$\Rightarrow \mathrm{P}(X\le 3)=0.445$

$\Rightarrow m\approx 3.94(3\text{ sf})$

$\Rightarrow \mathrm{P}(X=3)=\frac{\mathrm{e}^{-394}(394)^3}{3!}\approx 0.198$

$\Rightarrow \mathrm{P}(X<3)=0.445-0.198\approx 0.247$

7 **a** $\mathrm{P}=\mathrm{e}^{-\lambda}+\lambda\mathrm{e}^{-\lambda}+\frac{\lambda^2}{2!}\mathrm{e}^{-\lambda}$

b $\frac{\mathrm{dP}}{\mathrm{d}\lambda}=\frac{-\mathrm{e}^{-\lambda}}{2}(2+2\lambda+\lambda^2)+\frac{\mathrm{e}^{-\lambda}}{2}(2+2\lambda)$

$=\frac{\mathrm{e}^{-\lambda}}{2}(-2-2\lambda-\lambda^2+2+2\lambda)$

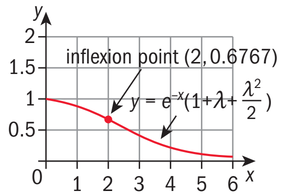
$=\frac{-\lambda^2\mathrm{e}^{-\lambda}}{2}$

λ^2 is always ≥ 0 and $\mathrm{e}^{-\lambda}$ is always > 0,

so $\frac{\mathrm{dP}}{\mathrm{d}\lambda}=\le 0.$

Hence $P(\lambda)$ is a decreasing function.

c



Exercise 10I

1 **a** discrete **b** continuous

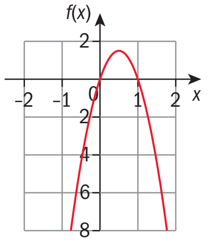
c continuous (sometimes treated as discrete)

d discrete as milk is bought in prepacked containers of fixed sizes

2 $f(x) \geq 0$ for all values of x and

$$\int_0^2 f(x)\,\mathrm{d}x = \int_0^2 \frac{1}{2} x\,\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^2$$
$$= 1 - 0 = 1$$

3 $f(x)$ has graph



a $\mathrm{E}(X) = \frac{1}{2}$ by symmetry

b $\mathrm{Var}(X) = \mathrm{E}(X^2) - \left(\frac{1}{2}\right)^2$

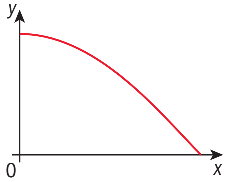
$$= \int_0^1 x^2 6x(1-x)\mathrm{d}x - \frac{1}{4}$$
$$= 6\int_0^1 (x^3 - x^4)\mathrm{d}x - \frac{1}{4}$$
$$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 - \frac{1}{4}$$
$$= 6 \times \left(\frac{1}{4} - \frac{1}{5}\right) - \frac{1}{4}$$
$$= \frac{6}{20} - \frac{1}{4} = \frac{1}{20}$$

c Median of $X = \frac{1}{2}$ by symmetry

d Mode of $X = \frac{1}{2}$ by symmetry.

Exercise 10J

1 **a**



b Median M is such that $\int_0^M 2\cos(2x)\mathrm{d}x = \frac{1}{2}$

$$\Rightarrow \sin 2M = \frac{1}{2}$$
$$\Rightarrow 2M = \frac{\pi}{6} \Rightarrow M = \frac{\pi}{12}$$

Mean $= \int_0^{\frac{\pi}{4}} x\,2\cos(2x)\mathrm{d}x$

$$= \left[x\sin(2x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin 2x\,\mathrm{d}x$$
$$= \frac{\pi}{4} + \left[\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{2}$$

Mode of $X = 0$

2 **a** $a = \int_0^1 x(1-2x+x^2)\mathrm{d}x = 1$

$$\Rightarrow a\int_0^1 (x - 2x^2 + x^3)\mathrm{d}x = 1$$
$$\Rightarrow a\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = 1$$
$$\Rightarrow a\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = 1 \Rightarrow a = 12$$

b $\mathrm{E}(X) = \int_0^1 x\,f(x)\mathrm{d}x = 12\int_0^1 x^2(1-2x+x^2)\mathrm{d}x$

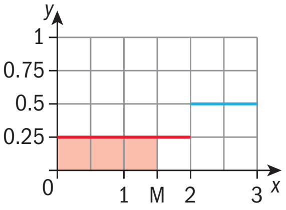
$$= 12\int_0^1 (x^2 - 2x^3 + x^4)\mathrm{d}x$$
$$= 12\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x^5}{5}\right]_0^1$$
$$= 12 \times \frac{1}{30} = \frac{2}{5}$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$$
$$= \int_0^1 x^2 f(x)\mathrm{d}x - \frac{4}{25}$$
$$= 12\int_0^1 x^3(1-2x+x^2)\mathrm{d}x - \frac{4}{25}$$
$$= 12\left[\frac{x^4}{4} - \frac{2x^5}{5} + \frac{x^6}{6}\right]_0^1 - \frac{4}{25}$$
$$= 12\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) - \frac{4}{25} = \frac{1}{25}$$

3 **a** $\int_0^2 k\,\mathrm{d}x + \int_2^3 2k\,\mathrm{d}x = 1$

$$\Rightarrow 2k + 2k = 1 \Rightarrow k = \frac{1}{4}$$

b



Median M given by $\int_0^M f(x)\mathrm{d}x = \frac{1}{2}$

i.e. shaded area $= \frac{1}{2}$ $\therefore Mk = \frac{1}{2}$

$$M = \frac{1}{2k} = 2$$

c $\mathrm{E}(X) = \int_0^3 x\,f(x)\,\mathrm{d}x = \int_0^2 kx\,\mathrm{d}x + \int_2^3 2kx\,dx$

$$= \left[\frac{kx^2}{2}\right]_0^2 + [kx^2]_2^3$$
$$= 2k + k(9-4) = 7k = \frac{7}{4} = 1\frac{3}{4}$$

$\mathrm{Var}(X) = \int_0^3 x^2 f(x)\mathrm{d}x - \frac{49}{16}$

$$= \int_0^2 kx^2\,\mathrm{d}x + \int_2^3 2kx^2\,\mathrm{d}x - \frac{49}{16}$$
$$= \left[\frac{kx^3}{3}\right]_0^2 + \left[\frac{2kx^3}{3}\right]_2^3 - \frac{49}{16}$$
$$= \frac{37}{48}$$

4 **a** $\int_0^3 (ax^2 + b)\mathrm{d}x = 1$

$$\Rightarrow \left[\frac{ax^3}{3} + bx\right]_0^3 = 1 \Rightarrow 9a + 3b = 1$$
$$\Rightarrow 3b = 1 - 9a \Rightarrow b = \frac{1-9a}{3}$$

b $\int_0^M f(x)dx = \frac{1}{2} \Rightarrow \int_0^1 (ax^2 + b)dx = \frac{1}{2}$

$\Rightarrow \left[\frac{ax^3}{3} + bx\right]_0^1 = \frac{1}{2}$

$\Rightarrow \frac{a}{3} + b = \frac{1}{2}$

Sub. in for b from part **a**

$\Rightarrow \frac{a}{3} + \frac{1-9a}{3} = \frac{1}{2}$

$\Rightarrow a + 1 - 9a = \frac{3}{2}$

$\Rightarrow 1 - \frac{3}{2} = 8a$

$\Rightarrow a = \frac{-1}{16}$

and $\therefore b = \frac{1+\frac{9}{16}}{3} = \frac{25}{48}$

c $\mathrm{E}(X) = \int_0^3 x f(x)\mathrm{d}x$

$= \int_0^3 \left(\frac{-1}{16}x^3 + \frac{25}{48}x\right)\mathrm{d}x$

$= \left[\frac{-x^4}{64} + \frac{25x^2}{96}\right]_0^3$

$= \frac{-81}{64} + \frac{75}{32} = 1\frac{5}{64}$

$\mathrm{Var}\ (X) = \int_0^3 x^2 f(x)\,\mathrm{d}x - \left(\frac{69}{64}\right)^2$

$= \int_0^3 \left(\frac{-1}{16}x^4 + \frac{25}{48}x^2\right)\mathrm{d}x - \left(\frac{69}{64}\right)^2$

$= \left[\frac{-x^5}{80} + \frac{25x^3}{144}\right]_0^3 - \left(\frac{69}{64}\right)^2$

$= \left(\frac{-243}{80} + \frac{675}{144}\right) - \left(\frac{69}{64}\right)^2 \approx 0.488$

5 **a** $\mathrm{E}(T) = \int_2^3 t f(t)\mathrm{d}t$

$= \int_2^3 \frac{6}{t}\mathrm{d}t$

$= [6\ln t]_2^3$

$= 6\ln 3 - 6\ln 2 = 6\ln\left(\frac{3}{2}\right).$

b $\mathrm{Var}(T) = \int_2^3 t^2 f(t)\mathrm{d}t - \left[6\ln\left(\frac{3}{2}\right)\right]^2$

$= \int_2^3 6\,\mathrm{d}t - \left[6\ln\left(\frac{3}{2}\right)\right]^2$

$= 6 - 36\left[\ln\left(\frac{3}{2}\right)\right]^2$

c M is such that $\int_2^M \frac{6}{t^2}\ \mathrm{d}t = \frac{1}{2}$

$\Rightarrow \left[\frac{-6}{t}\right]_2^M = \frac{1}{2}$

$\Rightarrow \frac{-6}{M} + 3 = \frac{1}{2}$

$\Rightarrow \frac{6}{M} = \frac{5}{2}$

$\Rightarrow M = \frac{2\times 6}{5} = \frac{12}{5} = 2\frac{2}{5}$

d Mode of T occurs where $f(t)$ is greatest

$f'(t) = \frac{-12}{t^3} =$ negative for $2 \le t \le 3$

$\therefore f(t)$ is decreasing for $2 \le t \le 3$

Or : Mode $= 2$

Exercise 10K

1 **a** $\mathrm{F}(x) = \int_0^x \frac{t^2}{9}\mathrm{d}t = \left[\frac{t^3}{27}\right]_0^x = \frac{x^3}{27} \quad (0 \le x \le 3)$

c $\mathrm{F}(x) = \int_0^x \cos t\,\mathrm{d}t = \sin x \left(0 \le x \le \frac{\pi}{2}\right)$

2 **a** $\mathrm{F}(x) = \int_0^x \frac{1}{4}t(4t^2)\mathrm{d}t$

$= \frac{1}{4}\int_0^x (4t - t^3)\mathrm{d}t$

$= \frac{1}{4}\left[2x^2 - \frac{x^4}{4}\right] = \frac{1}{16}(8x^2 - x^4)\ (0 \le x \le 2)$

b $\mathrm{P}\left(1 \le X \le \frac{3}{2}\right) = \mathrm{F}\left(\frac{3}{2}\right) - \mathrm{F}(1)$

$= \frac{1}{16}\left(8\times\frac{9}{4} - \frac{81}{16} - 8 + 1\right)$

$= \frac{1}{16}\left(10 - \frac{65}{16}\right) = \frac{95}{256}$

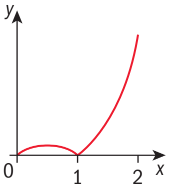
$\mathrm{P}(X > 1) = 1 - \mathrm{P}(X \le 1)$

$= 1 - \mathrm{F}(1)$

$= 1 - \frac{1}{16}(8-1)$

$= \frac{9}{16}$

3 **a**



b $k\int_0^2 (x^2 - x)\mathrm{d}x = 1$

$\Rightarrow k\int_0^1 (x^2 - x)\mathrm{d}x + k\int_1^2 (x^2 - x)\mathrm{d}x = 1$

$\Rightarrow k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 + k\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 = 1$

$\Rightarrow k\left(\frac{1}{2} - \frac{1}{3}\right) + k\left[\frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2}\right] = 1$

$\Rightarrow k\left(\frac{1}{6}\right) + k\left(\frac{5}{6}\right) = 1 \Rightarrow k = 1$

c $\mathrm{P}(1 \le X \le 2) = \int_1^2 k\left|x^2 - x\right|\mathrm{d}x$

$= \int_1^2 (x^2 - x)\mathrm{d}x$

$= \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_1^2 = \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2} = \frac{7}{3} - \frac{3}{2} = \frac{5}{6}$

4 **a** $\int_0^1 f(t)\,\mathrm{d}t = 1$

$\Rightarrow k\int_0^1 (\mathrm{e}-\mathrm{e}^{kt})\mathrm{d}t = 1$

$\Rightarrow k\left[\mathrm{e}t-\frac{1}{k}\mathrm{e}^{kt}\right]_0^1 = 1$

$\Rightarrow k\left[\mathrm{e}-\frac{1}{k}\mathrm{e}^{k}+\frac{1}{k}\right]=1$

$\Rightarrow \mathrm{e}k-\mathrm{e}^{k}+1=1$

$\Rightarrow \mathrm{e}^{k}=\mathrm{e}k$

$\Rightarrow \mathrm{e}^{k-1}=k$

$\Rightarrow k=1$ is the solution

b $\mathrm{P}\left(\frac{1}{3}<T<\frac{2}{3}\right)=\int_{\frac{1}{3}}^{\frac{2}{3}}(\mathrm{e}-\mathrm{e}^{t})\mathrm{d}t$

$=[\mathrm{e}t-\mathrm{e}^{t}]_{\frac{1}{3}}^{\frac{2}{3}}$

$=\mathrm{e}\frac{2}{3}-\mathrm{e}^{\frac{2}{3}}-\mathrm{e}\frac{1}{3}+\mathrm{e}^{\frac{1}{3}}$

$=\frac{1}{3}\mathrm{e}+\mathrm{e}^{\frac{1}{3}}-\mathrm{e}^{\frac{2}{3}}$

c $\mathrm{E}(T) = \int_0^1 x\,f(x)\,\mathrm{d}x$

$\int_0^1 (\mathrm{e}x-x\mathrm{e}^{x})\,\mathrm{d}x = \left[\frac{\mathrm{e}x^2}{2}\right]_0^1-\int_0^1 x\mathrm{e}^{x}\,\mathrm{d}x$

$=\frac{1}{2}\mathrm{e}-\left[x\mathrm{e}^{x}-\int \mathrm{e}^{x}\,\mathrm{d}x\right]_0^1$

$=\frac{1}{2}\mathrm{e}-[x\mathrm{e}^{x}-\mathrm{e}^{x}]_0^1$

$=\frac{1}{2}\mathrm{e}-\mathrm{e}+\mathrm{e}-1=\frac{1}{2}\mathrm{e}-1$

$\mathrm{Var}(T) = \int_0^1 x^2\,f(x)\mathrm{d}x-\left(\frac{1}{2}\mathrm{e}-1\right)^2$

$= \int_0^1 x^2\mathrm{e}\,\mathrm{d}x-\int_0^1 x^2\mathrm{e}^{x}\,\mathrm{d}x-\left(\frac{1}{2}\mathrm{e}-1\right)^2$

$=\left[\frac{x^3\mathrm{e}}{3}\right]_0^1-\left[x^2\mathrm{e}^{x}-\int 2x\mathrm{e}^{x}\,\mathrm{d}x\right]_0^1-\left(\frac{1}{2}\mathrm{e}-1\right)^2$

$=\frac{\mathrm{e}}{3}-[x^2\mathrm{e}^{x}]_0^1+2\int_0^1 x\mathrm{e}^{x}\,\mathrm{d}x-\left(\frac{1}{2}\mathrm{e}-1\right)^2$

$=\frac{\mathrm{e}}{3}-\mathrm{e}+2\left[x\mathrm{e}^{x}-\int \mathrm{e}^{x}\,\mathrm{d}x\right]_0^1-\left(\frac{1}{2}\mathrm{e}-1\right)^2$

$=\frac{-2}{3}\mathrm{e}+2[x\mathrm{e}^{x}-\mathrm{e}^{x}]_0^1-\left(\frac{1}{2}\mathrm{e}-1\right)^2$

$=\frac{-2}{3}\mathrm{e}+2-\left(\frac{1}{2}\mathrm{e}-1\right)^2$

$=\frac{-2}{3}\mathrm{e}+2-\frac{1}{4}\mathrm{e}^2+\mathrm{e}-1$

$=1+\frac{1}{3}\mathrm{e}-\frac{1}{4}\mathrm{e}^2$

d Prob. $= \int_{\frac{1}{2}}^{1}(\mathrm{e}-\mathrm{e}^{t})\mathrm{d}t=[\mathrm{e}t-\mathrm{e}^{t}]_{\frac{1}{2}}^{1}$

$=(\mathrm{e}-\mathrm{e})-\left(\frac{1}{2}\mathrm{e}-\mathrm{e}^{\frac{1}{2}}\right)$

$=\sqrt{\mathrm{e}}-\frac{1}{2}\mathrm{e}\approx 0.29$

5 **a** $\int_0^2 \frac{x}{4}\mathrm{d}x+\int_2^a \frac{5}{x^2}\mathrm{d}x=1$

$\Rightarrow \left[\frac{x^2}{8}\right]_0^2-\left[\frac{5}{x}\right]_2^a=1$

$\Rightarrow \frac{1}{2}-\frac{5}{a}+\frac{5}{2}=1$

$\Rightarrow 3=1+\frac{5}{a}$

$\Rightarrow \frac{5}{a}=2 \quad \Rightarrow a=\frac{5}{2}$

b CDF:

For $0\le x\le 2$, $\mathrm{F}(x)=\int_0^x \frac{t}{4}\,\mathrm{d}t=\frac{x^2}{8}$

For $2\le x\le 2\frac{1}{2}$, $\mathrm{F}(x)=\int_0^2 \frac{t}{4}\mathrm{d}t+\int_2^x \frac{5}{t^2}\mathrm{d}t$

$=\frac{1}{2}-\left[\frac{5}{t}\right]_2^x$

$=\frac{1}{2}-\frac{5}{x}+\frac{5}{2}$

$=3-\frac{5}{x}$

6 **a** $\int_{-3}^{3}\lambda(y+3)\mathrm{d}y=1$

$\Rightarrow \lambda\left[\frac{y^2}{2}+3y\right]_{-3}^{3}=1$

$\Rightarrow \lambda\left[\frac{\cancel{9}}{\cancel{2}}+9-\frac{\cancel{9}}{\cancel{2}}+9\right]=1$

$\Rightarrow 18\lambda=1$ or $\lambda=\frac{1}{18}$

b $f(y) = \int_{-3}^{y}\lambda(x+3)\mathrm{d}x$

$=\frac{1}{18}\left[\frac{x^2}{2}+3x\right]_{-3}^{y}$

$=\frac{1}{18}\left(\frac{y^2}{2}+3y-\frac{9}{2}+9\right)$

$=\frac{1}{36}(y^2+6y+9)=\frac{1}{36}(y+3)^2$

c $\mathrm{P}(0\le Y\le 1)=\mathrm{F}(1)-\mathrm{F}(0)=\frac{16}{36}-\frac{9}{36}=\frac{7}{36}$

$\mathrm{P}(Y>1)=1-\mathrm{P}(Y\le 1)=1-\mathrm{F}(1)=1-\frac{16}{36}=\frac{5}{9}$

d $\mathrm{E}(Y)=\int_{-3}^{3} y\,f(y)\mathrm{d}y$

$=\frac{1}{18}\int_{-3}^{3}(y^2+3y)\mathrm{d}y=\frac{1}{18}\left[\frac{y^3}{3}+\frac{3y^2}{2}\right]_{-3}^{3}$

$=\frac{1}{18}\left[9+\frac{\cancel{27}}{\cancel{2}}+9-\frac{\cancel{27}}{\cancel{2}}\right]=1$

$\mathrm{E}(Y^2)=\frac{1}{18}\int_{-3}^{3}(y^3+3y^2)\mathrm{d}y$

$=\frac{1}{18}\left[\frac{y^4}{4}+y^3\right]_{-3}^{3}$

$=\frac{1}{18}\left[\frac{81}{4}+27-\frac{81}{4}+27\right]$

$=\frac{54}{18}=3$

$\therefore \mathrm{Var}(Y)=3-1^2=2$

e $\frac{1}{36}(y+3)^2 = \frac{1}{4}$

$\Rightarrow (y+3)^2 = 9$

$\Rightarrow y+3 = \pm 3$

$\Rightarrow y = 0 \text{ or } -6(\text{impossible})$

$\therefore y = 0$ (the lower quartile).

7 $\int_1^4 (ax^2+bx+c)\,dx = 1$

$\Rightarrow \left[\frac{ax^3}{3}+\frac{bx^2}{2}+cx\right]_1^4 = 1$

$\Rightarrow a\frac{64-1}{3}+b\frac{16-1}{2}+c\times 3 = 1$

$\Rightarrow 21a+\frac{15b}{2}+3c = 1 \qquad (1)$

Mode $= 2 \Rightarrow 2ax+b = 0$ when $x = 2$

$\Rightarrow 4a+b = 0$

$\Rightarrow b = -4a \qquad (2)$

$\text{E}(X) = 3 \Rightarrow \int_1^4 (ax^3+bx^2+cx)\,dx = 3$

$\Rightarrow \left[\frac{ax^4}{4}+\frac{bx^3}{3}+\frac{cx^2}{2}\right]_1^4 = 3$

$\Rightarrow 64a+\frac{64b}{3}+8c-\frac{a}{4}-\frac{b}{3}-\frac{c}{2} = 3$

$\Rightarrow \frac{255a}{4}+\frac{63b}{3}+\frac{15c}{2} = 3$

$\Rightarrow \frac{255a}{4}+21b+\frac{15c}{2} = 3$

$\Rightarrow 255a+84b+30c = 12 \qquad (3)$

Sub (2) into (1) $\Rightarrow 21a-\frac{60a}{2}+3c = 1$

$\Rightarrow -9a+3c = 1 \qquad (4)$

Sub (2) into (3)

$\Rightarrow 255a-336a+30c = 12$

$$\begin{array}{rr} & -\ 81a+30c = 12 \\ a\times 4 \ \Rightarrow & -\ 81a+27c = \ 9 \\ \hline & 3c = 3 \end{array}$$

$c = 1 \quad \Rightarrow \quad a = \frac{2}{9}$ from (4)

$\therefore b = \frac{-8}{9}$ from (2)

8 **a** $\alpha\int_{\frac{\pi}{2}}^{\frac{\sqrt{\pi}}{2}} (1+\sin x)\,dx = 1 \quad \Rightarrow \alpha[x-\cos x]_{\frac{\pi}{2}}^{\frac{\sqrt{\pi}}{2}} = 1$

$\Rightarrow \alpha\left(\frac{\sqrt{\pi}}{2}-0-\frac{\pi}{2}+0\right) = 1$

$\Rightarrow 2\pi\alpha = 1 \Rightarrow \alpha = \frac{1}{2\pi}$

b $\text{P}(X<\pi) = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2\pi}(1+\sin x)\,dx$

$= \frac{1}{2\pi}[x-\cos x]_{\frac{\pi}{2}}^{\pi}$

$= \frac{1}{2\pi}\left[\pi+1-\frac{\pi}{2}+0\right]$

$= \frac{1}{2\pi}\left[\frac{\pi}{2}+1\right] = \frac{1}{4}+\frac{1}{2\pi}$

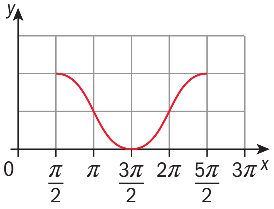
$\text{P}(X<2\pi)\int_{\frac{\pi}{2}}^{2\pi} \frac{1}{2\pi}(1+\sin x)\,dx$

$= \frac{1}{2\pi}\left[2\pi-\cos 2\pi-\frac{\pi}{2}+0\right]$

$= \frac{1}{2\pi}\left[\frac{3\pi}{2}-1\right]$

$= \frac{3}{4}-\frac{1}{2\pi}$

c



Median $= \frac{3\pi}{2}$

d $\text{F}(x) = \int_{\frac{\pi}{2}}^{x} f(t)\,dt$

$= \frac{1}{2\pi}[t-\cos t]_{\frac{\pi}{2}}^{x}$

$= \frac{1}{2\pi}\left[x-\cos x-\frac{\pi}{2}+0\right]$

$= \frac{x}{2\pi}-\frac{1}{2\pi}\cos x-\frac{1}{4}$

e $\text{F}(a) = 0.75$

$\Rightarrow \frac{a}{2\pi}-\frac{1}{2\pi}\cos a-\frac{1}{4} = \frac{3}{4}$

$\Rightarrow a-\cos a = 2\pi$

$\Rightarrow a = 7.0\,(1\text{ dp})$

Similarly for b, $\quad \text{F}(b) = 0.25$

$\Rightarrow b-\cos b = \pi$

$b = 2.4\,(1\text{ dp})$

$\Rightarrow \text{IQR} = a-b \approx 4.6$

Exercise 10L

1 **a** $\text{P}(0<X<1.5) = \text{P}\left(\frac{0-1}{2}<Z<\frac{1.5-1}{2}\right)$

$= \text{F}(0.25)-\text{F}(-0.5)$

$= \text{F}(0.25)-(1-\text{F}(0.5))$

$= 0.5987+0.6915-1$

≈ 0.290

b $\text{P}(X<0.5) = \text{P}\left(Z<\frac{0.5-1}{2}\right)$

$= \text{P}\left(Z<-\frac{1}{4}\right) = \text{F}(-0.25)$

$= 1-\text{F}(0.25) \approx 0.401$

c $\text{P}(X\geq 3) = \text{P}\left(Z\geq\frac{3-1}{2}\right)$

$= \text{P}(Z\geq 1)$

$= 1-\text{F}(1) \approx 0.159$

2 **a** $\text{P}(X<45) = \text{P}\left(Z<\frac{45-50}{20}\right)$

$= \text{F}(-0.25)$

$= 1-\text{F}(0.25) \approx 0.401$

b $P(37 \le X < 65) = P\left(\frac{37-50}{20} \le Z < \frac{65-50}{20}\right)$

$= F(0.75) - F(-0.65) = F(0.75) - 1 + F(0.65)$

≈ 0.516

c $P(X \ge 52) = P\left(Z \ge \frac{52-50}{20}\right) = P(Z \ge 0.1)$

$= 1 - F(0.1)$

≈ 0.460

3 **a** Mean = 35, SD = 7

b $P(X < 25) = P\left(Z < \frac{25-35}{7}\right) \approx P(Z < -1.43)$

$= 1 - F(1.43) \approx 0.076$

$P(29 \le X \le 41) = P\left(\frac{29-35}{7} \le X \le \frac{41-35}{7}\right)$

$\approx P(-0.86 \le X \le 0.86)$

$= F(0.86) - F(-0.86)$

$= F(0.86) - 1 + F(0.86)$

≈ 0.610

$P(X \ge 45) = P\left(Z \ge \frac{45-35}{7}\right)$

$\approx P(Z \ge 1.43)$

$= 1 - F(1.43)$

≈ 0.076

Exercise 10M

1 $X \sim N\left(150, 0.5^2\right)$

a $P(X < 149) = P\left(Z < \frac{149-150}{0.5}\right)$

$= P(Z > -2)$

$= 1 - P(Z < 2)$

≈ 0.023

b $P(X > 151.5) = P\left(Z < \frac{151.5-150}{0.5}\right)$

$= P(Z < -3)$

$= 1 - P(Z < 3)$

≈ 0.001

c $P(149 < X < 151) = P(-2 < Z < 2)$

$= F(2) - F(-2)$

$= F(2) - (1 - F(-2))$

$= 2F(2) - 1$

≈ 0.9544

2 **a** $M \sim N\left(1.1, 0.15^2\right) \quad (\mu = 1.1, \sigma = 0.15)$

b $P(1.2 < M < 1.3) = P\left(\frac{1.2-1.1}{0.15} < Z < \frac{1.3-1.1}{0.15}\right)$

$= F(1.33) - F(0.67)$

$= 0.9082 - 0.7486$

≈ 0.160

c $P(M > 1.4) = P\left(Z > \frac{1.4-1.1}{0.15}\right) = P(Z > 2)$

$= 1 - F(2) = 0.0228$

Estimated no. of cabbages $= 0.0228 \times 850 \approx 19$

3 $T \sim N\left(20.4, 3.5^2\right)$

a $P(T < 18.1) = P\left(Z < \frac{18.1-20.4}{3.5}\right) = P(Z < -0.66)$

$= 1 - F(0.66)$

≈ 0.255

$P(T > 17.9) = P\left(Z > \frac{17.9-20.4}{3.5}\right) = P(Z > -0.71)$

$= F(0.71)$

≈ 0.761

b $P\left(T < 18.1 \middle| T > 17.9\right) = \frac{P(17.9 < T < 18.1)}{P(T > 17.9)}$

$= \frac{0.7611 - 0.2546}{0.7611}$

≈ 0.665

c $P(T < t) = 0.444$

$\Rightarrow P\left(Z < \frac{t-20.4}{3.5}\right) = 0.444$

$\Rightarrow F\left(\frac{t-20.4}{3.5}\right) = 0.444$

$\Rightarrow \frac{t-20.4}{3.5} = -0.14$

$\Rightarrow t = 20.4 - 0.14 \times 3.5$

≈ 19.9

Exercise 10N

1 $X \sim N\left(5, \sigma^2\right) \qquad P(X < 3) = 0.3$

a $P(X \ge 7) = P(X \le 3) = 0.3$

b $P(X < 7) = 1 - 0.3 = 0.7$

c $P(3 \le X \le 7) = 0.7 - 0.3 = 0.4$

2 $Y \sim N\left(12, \sigma^2\right) \qquad P(10 \le Y < 14) = 0.6$

a $P(Y \ge 14) = \frac{1}{2}(1 - 0.6) = 0.2$

b $P(Y < 10) = 0.2$

c $P(12 \le Y < 14) = 0.8 - 0.5 = 0.3$

d $P\left(Y < 14 \middle| Y > 12\right) = \frac{P(12 < Y < 14)}{P(Y > 12)} = \frac{0.3}{0.5} = 0.6$

3 $X \sim N\left(-5, \sigma^2\right) \qquad P(X < -3) = 0.8$

a $P(X < -7) = 0.2$

b $P(-7 \le X < -5) = 0.5 - 0.2 = 0.3$

c $P(X < -7) + P(X > -3) = 0.2 + 0.2 = 0.4$

4 **a** Mean = 10, SD = 5

b $P(X < 5) = P\left(Z < \frac{5-10}{5}\right) = P(Z < -1)$

$= 1 - F(1) \approx 0.159$

$P(X \ge 15) = P\left(Z > \frac{15-10}{5}\right) = P(Z > 1) = 0.159$

c $F(a) = 0.223 \Rightarrow \frac{a-10}{5} = -0.76$

$\Rightarrow a = 10 - 5 \times 0.76 \approx 6.2$

$\Rightarrow b = 10 + (10 - 6.2) = 13.8$

Exercise 10O

1 **a** 0.1151, 0.8849 **b** 0.0107, 0.9893

c 0.0047, 0.9953

They add to 1 in each case

2 **a** –0.525, 0.525 **b** –0.253, 0.253

They are equidistant either side of the mean

3 **a** 0.6736, –0.124 **b** $a = 0$, $b = 0.6915$

4 **a** They are both the same and equal 0.8413

b $P(0 \le X \le 2) = P\left(\frac{-1}{2} \le Z \le \frac{1}{2}\right) = 0.383$

5 $X \sim \mathrm{N}(-6, 3^2)$ $\quad Z \sim \mathrm{N}(0, 1)$

$$P(0 \le Z \le 1.5) = \Phi(1.5) - \Phi(0) = 0.9332 - 0.5 = 0.4332$$

$$P(-6 \le X \le -1.5) = P(0 \le Z \le 1.5) = 0.4332$$

Exercise 10P

1 $\mathrm{P}\left(Z < \frac{5-\mu}{9}\right) = 0.754$

$\Rightarrow \frac{5-\mu}{9} \approx 0.69 \Rightarrow \mu \approx -1.2$

2 $\mathrm{P}\left(Z \le \frac{1-\mu}{\sigma}\right) = 0.345$ and $\mathrm{P}\left(Z \le \frac{3-\mu}{\sigma}\right) = 0.943$

$\Rightarrow \frac{1-\mu}{\sigma} \approx -0.40$ and $\frac{3-\mu}{\sigma} \approx 1.58$

$\Rightarrow 1 = \mu - 0.40\sigma \quad 1$

$3 = \mu + 1.58\sigma \quad 2$

$\Rightarrow 2 = 1.98\sigma \Rightarrow \sigma \approx 1.0$ and $\mu \approx 1.40$

3 $\mathrm{P}(X > 58.44) = 0.022 \Rightarrow \mathrm{P}(X \le 58.44) = 0.978$

$\Rightarrow \Phi\left(\frac{58.44 - M}{\sigma}\right) = 0.978$

$\Rightarrow \frac{58.44 - M}{\sigma} \approx 2.01$

$\Rightarrow \mu = 58.44 - 2.01\sigma \quad (1)$

$\mathrm{P}(X < 48.84) = 0.012 \Rightarrow \Phi\left(\frac{48.48\mu}{\sigma}\right) = 0.012$

$\Rightarrow \frac{48.84\mu}{\sigma} = -2.26$

$\Rightarrow 48.84 + 2.26\sigma = \mu \quad (2)$

$\Rightarrow 48.84 + 2.26\sigma = 58.44 - 2.01\sigma \ (5)$

$\Rightarrow 4.27\sigma = 9.6$

$\Rightarrow \sigma \approx 2.25$

Hence $\mu \approx 53.9$

4 $X \sim \mathrm{N}(1.03, \sigma^2)$

$\Rightarrow \mathrm{P}(X < 1) = 0.018$

$\Phi\left(\frac{1 - 1.03}{\sigma}\right) = 0.018$

$\Rightarrow \frac{1 - 1.03}{\sigma} = -2.10$

$\Rightarrow \sigma = \frac{0.03}{2.10} \approx 0.0143\,\mathrm{kg} = 14.3\,\mathrm{grams}$

5 **a** $\Phi\left(\frac{50.1 - \mu}{\sigma}\right) = 1 - 0.119 = 0.881$

$\Rightarrow \frac{50.1 - \mu}{\sigma} = 1.18 \Rightarrow 50.1 = \mu + 1.18\sigma$

$\Phi\left(\frac{43.6 - \mu}{\sigma}\right) = 0.305 \Rightarrow \frac{43.6 - \mu}{\sigma} = -0.51 \quad (1)$

$\Rightarrow 43.6 = \mu - 0.51\sigma \quad (2)$

$(1) - (2) \Rightarrow 6.5 = 1.69\sigma \Rightarrow \sigma = 3.85$ or $\mu \approx 45.6$

b $\mathrm{P}\left(|X - \mu| < \frac{\sigma}{2}\right)$

$= \mathrm{P}\left(\frac{-\sigma}{2} < X - \mu < \frac{\sigma}{2}\right)$

$= \mathrm{P}\left(-\frac{1}{2} < \frac{X - \mu}{\sigma} < \frac{1}{2}\right)$

$= \Phi\left(\frac{1}{2}\right) - \Phi\left(\frac{-1}{2}\right)$

$= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{2}\right)\right)$

$= 2\Phi\left(\frac{1}{2}\right) - 1$

≈ 0.383

6 $\mathrm{P}(X \ge 80) = 0.1 \Rightarrow \mathrm{P}(X < 80) = 0.9$

$\Rightarrow \Phi\left(\frac{80 - \mu}{\sigma}\right) = 0.9$

$\Rightarrow \frac{80 - \mu}{\sigma} = 1.282$

$\Rightarrow 80 = \mu + 1.282\sigma \quad (1)$

$\mathrm{P}(X < 45) = 0.2$

$\Rightarrow \Phi\left(\frac{45 - \mu}{\sigma}\right) = 0.2$

$\Rightarrow \frac{45 - \mu}{\sigma} = -0.842$

$\Rightarrow 45 = \mu - 0.842\sigma \quad (2)$

$(1) - (2) \Rightarrow 35 = 2.124\sigma$

$\Rightarrow \sigma = 16.5$

$\mu = 58.9$

7 $X \sim \mathrm{N}(550, 20)$

a $\mathrm{P}(500 < X < 600) = \mathrm{P}\left(\frac{500 - 550}{20} < z < \frac{600 - 550}{20}\right)$

$= \mathrm{P}\left(-2\frac{1}{2} < z < 2\frac{1}{2}\right) = \Phi\left(2\frac{1}{2}\right) - \Phi\left(-2\frac{1}{2}\right)$

$= 2\Phi\left(2\frac{1}{2}\right) - 1 \approx 0.988$

b $\mathrm{P}(X > M) = 0.1 \Rightarrow (X \le M) = 0.9$

$\Rightarrow \Phi\left(\frac{M - 500}{20}\right) = 0.9$

$\Rightarrow \frac{M - 550}{20} = 1.282$

$\Rightarrow M = 576\,\mathrm{grams}$

c $\mathrm{P}(X > 540) = \mathrm{P}\left(z > \frac{540 - 550}{20}\right) = \mathrm{P}\left(z > \frac{-1}{2}\right)$

$= \mathrm{P}\left(z < \frac{1}{2}\right) = 0.6915$

$\therefore$ Estimated number $= 1200 \times 0.6915 \approx 830$

d $M \sim \mathrm{N}(\mu, \sigma^2)$

$$\mathrm{P}(M \geq 600) = 0.15 \Rightarrow P(M < 600) = 0.85$$

$$\Rightarrow \Phi\left(\frac{600-\mu}{\sigma}\right) = 0.85$$

$$\frac{600-\mu}{\sigma} = 1.037$$

$$\Rightarrow 600 = \mu + 1.037\sigma \quad (1)$$

$$\mathrm{P}(M < 540) = 0.1$$

$$\Rightarrow \Phi\left(\frac{540-\mu}{\sigma}\right) = 0.1$$

$$\Rightarrow \frac{540-\mu}{\sigma} = -1.282$$

$$\Rightarrow 540 = \mu - 1.282\sigma \quad (2)$$

$$(1)-(2) \Rightarrow 60 = 2.319\sigma$$

$$\Rightarrow \sigma = 25.9 \text{ grams}$$

$$\mu = 573 \text{ grams}$$

8 $M \sim N\,(1.02, \sigma^2)$

$$\mathrm{P}(M < 1) < 0.01$$

$$\Rightarrow \mathrm{P}\left(Z < \frac{1-0.02}{\sigma}\right) < 0.01$$

Review exercise

1 **a** $\sum \mathrm{P}(X) = 1$

$$\Rightarrow 2a^2 + 3a + 3a^2 + 2a + 2a^2 + a = 1$$

$$\Rightarrow 7a^2 + 6a - 1 = 0$$

$$\Rightarrow (7a-1)(a+1) = 0$$

$$\Rightarrow a = \frac{1}{7}$$

(a cannot be negative as P(2) would then be negative)

b PDF is:

x	1	2	3	4
$\mathrm{P}(X=x)$	$\frac{2}{49}$	$\frac{21}{49}$	$\frac{17}{49}$	$\frac{9}{49}$

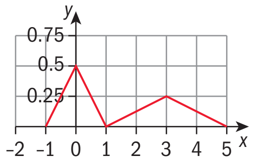
$$\text{Mean} = \sum x\mathrm{P}(x) = \frac{2}{49} + \frac{42}{49} + \frac{51}{49} + \frac{36}{49}$$

$$= \frac{131}{49} = 2\frac{33}{49}$$

Mode = 2

Median = 3

2 **a**



Total area = 1, so $\frac{1}{2} \times 4 \times \frac{2}{k} + \frac{1}{2} \times 2 \times \frac{1}{2} = 1$

$$\Rightarrow \frac{4}{k} + \frac{1}{2} = 1 \Rightarrow \frac{4}{k} = \frac{1}{2}$$

$$\Rightarrow k = 8$$

b Median = 1, Mode = 3

c $$\mathrm{P}(0 \leq X \leq 3 | X \geq 1) = \frac{\mathrm{P}(1 \leq X \leq 3)}{\mathrm{P}(X \geq 1)}$$

$$= \frac{\frac{1}{2} \times 2 \times \frac{1}{4}}{\frac{1}{2}}$$

$$= \frac{1}{2}$$

3 Let X = number graduating $X \sim \mathrm{B}(n, p), n = 5, p = \frac{4}{5}$

a $\mathrm{P}(X=0) = \left(\frac{1}{5}\right)^5 = \frac{1}{3125}$

b $\mathrm{P}(X=5) = \left(\frac{4}{5}\right)^5 = \frac{1024}{3125}$

c $\mathrm{P}(X \geq 2) = 1 - \mathrm{P}(0) - \mathrm{P}(1)$

$$= 1 - \left(\frac{1}{5}\right)^5 - 5 \times \left(\frac{1}{5}\right)^4 \times \frac{4}{5}$$

$$= 1 - \frac{21}{3125}$$

$$= \frac{3104}{3125}$$

4 Let X = the no. correct on the last 5 questions

Then $X \sim \mathrm{B}(n, p)$ where $n = 5$, $p = \frac{1}{2}$

a $\mathrm{P}(X=5) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

b $\mathrm{E}(X) = np = 2\frac{1}{2}$

If get X correct Y is the final score, than

$$\mathrm{Y} = 5 \times 2 + X \times 2 + (5 - X) \times -1$$

$$= 10 + 2 \times -5 + X$$

$$5 + 3X$$

$\therefore$ Expected final score is $\mathrm{E}(Y) = 5\mathrm{E}(X) + 3 = 5 \times 2.5 + 3$

$$= 12.5$$

It is obviously worth guessing the last 5, because it he stops after the first 5, he will only get a score of 10.

5 **a** $\mathrm{E}(T) = \mathrm{Var}(T) = m$

But $\mathrm{Var}(T) = \mathrm{E}(T^2) - (\mathrm{E}(T))^2$

$$\therefore \ m = 6 - m^2$$

$$\Rightarrow m^2 + m - 6 = 0$$

$$\Rightarrow (m+3)(m-2) = 0$$

$$\Rightarrow m = -3 \quad or \quad 2$$

$m = -3$ is impossible as it would give

$P(0) = \mathrm{e}^3 > 1$

$$\therefore m = 2$$

b $P(X=0) = \frac{\mathrm{e}^{-m} m^0}{0!} = \mathrm{e}^{-2} = 0.135\left(\text{or } \frac{1}{\mathrm{e}^2}\right)$

6 **a** $a = 1$

b $\mathrm{F}(m) = \frac{1}{2} \Rightarrow \tan m = \frac{1}{2} \Rightarrow m \approx 0.46\left(\text{or } \tan^{-1}\left(\frac{1}{2}\right)\right)$

c $f(x) = F'(x) = \begin{cases} \sec^2 x & 0 \le x \le \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$

d $P\left(x \le \frac{\pi}{4}\right) = F\left(\frac{\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

Review exercise

1 Let X = width of a piece

Then $X \sim N(20.05, 0.02^2)$

a $\Rightarrow P(20.02 < X < 20.06)$

$= P\left(\frac{20.02 - 20.05}{0.02} < z < \frac{20.06 - 20.05}{0.02}\right)$

$= P(-1.5 < t < 0.5) = F(0.5) - F(-1.5)$

$= 0.691462 - 0.066807$

≈ 0.625

b $P(X < 20.00) = P\left(z < \frac{20.00 - 20.05}{0.02}\right) = F(-2.5)$

$\simeq 0.0062$

2. Let X = life of a motor and G the guarantee period.

Then X is $N(15, 2^2)$ and we require:

$P(X \le G) = 0.001$

i.e. $P\left(z \le \frac{G - 15}{2}\right) = 0.001$

$\Rightarrow \frac{G - 15}{2} \approx -3.10$

$\Rightarrow G \cong 15 - 6.2$

≈ 8.8

(Should only give an 8 year guarantee)

3 **a** $\int_0^{\sqrt{2}} \frac{k}{2 + x^2} dx = 1 \Rightarrow k\left[\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\sqrt{2}} = 1$

$\Rightarrow \frac{k}{\sqrt{2}}\left[\frac{\pi}{4}\right] = 1 \Rightarrow k = \frac{4\sqrt{2}}{\pi} \approx 1.8$

b $P\left(X \le \frac{1}{2}\right) = \int_0^{\frac{1}{2}} f(x)\, dx = \frac{k}{\sqrt{2}}\left[\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\frac{1}{2}}$

$= \frac{4}{\pi} \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

≈ 0.433

c $E(X) = \int_0^{\frac{1}{2}} x f(x)\, dx = k \int_0^{\frac{1}{2}} \frac{x}{2 + x^2} dx$

$= k\left[\frac{1}{2} \ln(2 + x^2)\right]_0^{\frac{1}{2}} = \frac{k}{2}(\ln 2.25 - \ln 2)$

$= \frac{k}{2} \ln\left(\frac{2.25}{2}\right)$

≈ 0.106

4 $X \sim P_0(4)$ $(m = 4 \text{ in 2hrs})$

a $P(X \ge 1) = 1 - P(0) = 1 - e^{-m}$

≈ 0.982

b $P(X > 5) = 1 - P(X \le 5) \approx 0.215$

5 **a** $a\int_1^3 \frac{1}{x(4 - x)} dx = 1$

$\Rightarrow \frac{a}{4}\left[\ln\left(\frac{x}{4 - x}\right)\right]_1^3 = 1 \Rightarrow \frac{a}{4}\left[\ln 3 - \ln\frac{1}{3}\right] = 1$

$\Rightarrow \frac{a}{4} \ln 9 = 1 \Rightarrow a = \frac{4}{\ln 9} \approx 1.82048$

b $E(X) = \int_1^3 x f(x) dx$

$= a \int_1^3 \frac{1}{4 - x} dx$

$= a[-\ln(4 - x)]_1^3$

$= \frac{4}{\ln 9} \times \ln 3$

$= \frac{4}{2} = 2$

$Var(X) = E(X^2) - 2^2$

$= \int_1^3 x^2 f(x)\, dx - 4$

$= a \int_1^3 \frac{x}{4 - x} dx - 4$

$= a \int_1^3 -1 + \frac{4}{4 - x} dx - 4$

$= a[-x - 4\ln(4 - x)]_1^3 - 4$

$= a[-3 - 4\ln 1 + 1 + 4\ln 3] - 4$

$= a[-2 + 4\ln 3] - 4$

$= \frac{4}{\ln 9} \times -2 + \frac{16}{2} - 4$

$= \frac{-8}{\ln 9} + 4 \approx 0.35904$

c $P(X < 2) = \int_1^2 f(x)\, dx = \left[\frac{a}{4} \ln\left(\frac{x}{4 - x}\right)\right]_1^2$

$= \frac{a}{4}\left(\ln 1 - \ln\left(\frac{1}{3}\right)\right)$

$= \frac{1}{\ln 9} \times \ln 3 = \frac{1}{2}$

12 **a** $250 \times \frac{71}{160} = 110\frac{15}{16} \approx 111$ customers

b **i** X = no. of customers who buy vanilla

Then $X \sim B\left(5, \frac{3}{16}\right)$

$\therefore P(X = 3) = 5c_3\left(\frac{3}{16}\right)^3 \left(\frac{13}{16}\right)^2$

≈ 0.0435

ii Y = not buying ice cream.

$Y \sim B\left(5, \frac{3}{160}\right)$

$P(Y = 2) = 5c_2\left(\frac{3}{160}\right)^2 \left(\frac{157}{160}\right)^3 \approx 0.0033$

c $2 \times \frac{14}{160} \times \frac{1}{160} \approx 0.0011$

Have assumed the 2 choices of flavor are independent of one another.

11

Inspiration and formalism

Answers

Skills check

1 P(–1, 5) Q(2, 1) $PQ^2 = (-1-2)^2 + (5-1)^2 = 25$

$PQ = 5$

2 A(1, 3) B(4, 9)

a gradient $= \frac{9-3}{4-1} = 2$

b $y - 3 = 2(x - 1) \Rightarrow y = 2x + 1$

Exercise 11A

1 **a** opposite sides of a regular hexagon are equal and parallel $\therefore\ \overrightarrow{AB} = \overrightarrow{ED}$

b **i** $\overrightarrow{FC}$ and $\overrightarrow{ED}$

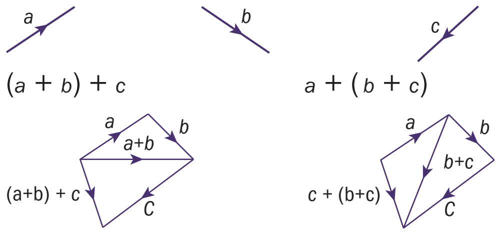
ii $\overrightarrow{AD}$, $\overrightarrow{DA}$, $\overrightarrow{BE}$, $\overrightarrow{EB}$ and $\overrightarrow{FC}$

2 **a** No, 2 sides of a pentagon are parallel $\therefore$ the vectors in the diagram are all district.

b No, since no 2 vectors in the diagram are parallel and equal in length

Exercise 11B

1



2 **a** $\overrightarrow{AF} + \overrightarrow{BC} = \overrightarrow{AF} + \overrightarrow{FE} = \overrightarrow{AE}$

b $\frac{1}{2}\overrightarrow{AD} + \overrightarrow{ED} = \overrightarrow{FE} + \overrightarrow{ED} = \overrightarrow{FD}$

c $2\overrightarrow{FE} - \overrightarrow{AF} - \overrightarrow{FE} = \overrightarrow{FE} + \overrightarrow{FA}$

$= \overrightarrow{FE} + \overrightarrow{EO} = \overrightarrow{FO} = \overrightarrow{AB}$

d $\frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BE}) = \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{BE} = \overrightarrow{AO} + \overrightarrow{OE} = \overrightarrow{AE}$

e $-\frac{1}{2}\overrightarrow{FC} + \overrightarrow{BC} = \overrightarrow{OF} + \overrightarrow{FE} = \overrightarrow{OE} = \overrightarrow{CD}$

f $-2\overrightarrow{ED} - \overrightarrow{AF} + \overrightarrow{AB} = \overrightarrow{CF} + \overrightarrow{FA} + \overrightarrow{AB} = \overrightarrow{CB}$

(other answers are possible in this question)

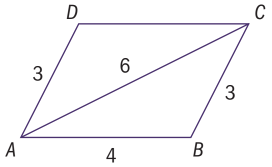
3 **a** **i** $\overrightarrow{AC} = \mathbf{u} + \mathbf{v}$

ii $\overrightarrow{HB} = \mathbf{u} - \mathbf{w} + \mathbf{u} = 2\mathbf{u} - \mathbf{w}$

iii $\overrightarrow{CE} = -\mathbf{v} - \mathbf{u} + \mathbf{w} - \mathbf{u} - \mathbf{v} = \mathbf{w} - 2\mathbf{u} - 2\mathbf{v}$

iv $\overrightarrow{AF} = \mathbf{w} - \mathbf{v}$

b **i**



$\cos B = \frac{4^2 + 3^2 - 6^2}{2 \times 4 \times 3} = -\frac{11}{24}$

$A\hat{B}C = 117^{\circ}$

ii area ABCD $= 2\left(\frac{1}{2} \times 4 \times 3 \times \sin\ 117°\right)$

$= 10.7$ sq. units

4 **a** $3\mathbf{x} - \mathbf{u} = 6\mathbf{v} + 2\mathbf{u} \Rightarrow 3\mathbf{x} = 6\mathbf{v} + 3\mathbf{u}$

$\Rightarrow \mathbf{x} = 2\mathbf{v} + \mathbf{u}$

b $2(\mathbf{x} - \mathbf{u}) + 3(\mathbf{u} - \mathbf{v}) = 0 \Rightarrow 2\mathbf{x} - 2\mathbf{u} + 3\mathbf{u} - 3\mathbf{v} = 0$

$\mathbf{x} = \frac{1}{2}(3\mathbf{v} - \mathbf{u})$

c $\frac{1}{2}(\mathbf{x} - \mathbf{u}) = \frac{1}{3}(\mathbf{x} + \mathbf{v}) \Rightarrow 3\mathbf{x} - 3\mathbf{u} = 2\mathbf{x} + 2\mathbf{v}$

$\Rightarrow \mathbf{x} = 3\mathbf{u} + 2\mathbf{v}$

Exercise 11C

1 A(–1, 3) C(5, 4) I(7, 8)

a **i** $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\left[\begin{pmatrix}5\\4\end{pmatrix} - \begin{pmatrix}-1\\3\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}6\\1\end{pmatrix} = \begin{pmatrix}3\\0.5\end{pmatrix}$

ii $\overrightarrow{AE} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{CI}$

$= \begin{pmatrix}3\\0.5\end{pmatrix} + \frac{1}{2}\left[\begin{pmatrix}7\\8\end{pmatrix} - \begin{pmatrix}5\\4\end{pmatrix}\right] = \begin{pmatrix}4\\2.5\end{pmatrix}$

iii $\overrightarrow{CD} = \frac{1}{2}\overrightarrow{CI} = \begin{pmatrix}1\\2\end{pmatrix}$

b **i** $\overrightarrow{BF} = \overrightarrow{BA} + \overrightarrow{AF} = \begin{pmatrix}-3\\-0.5\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = -2\mathbf{i} + 1.5\mathbf{j}$

ii $\overrightarrow{CH} = \overrightarrow{CB} + \overrightarrow{BH} = \begin{pmatrix}-3\\-0.5\end{pmatrix} + \begin{pmatrix}2\\4\end{pmatrix} = -\mathbf{i} + 3.5\mathbf{j}$

iii $\overrightarrow{DG} = \overrightarrow{DF} + \overrightarrow{FG} = \begin{pmatrix}-6\\-1\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = -5\mathbf{i} + \mathbf{j}$

c $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \begin{pmatrix}-1\\3\end{pmatrix} + \begin{pmatrix}3\\0.5\end{pmatrix} = \begin{pmatrix}2\\3.5\end{pmatrix}$

$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = \begin{pmatrix}5\\4\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}6\\6\end{pmatrix}$

$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \begin{pmatrix}-1\\3\end{pmatrix} + \begin{pmatrix}4\\2.5\end{pmatrix} = \begin{pmatrix}3\\5.5\end{pmatrix}$

$\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF} = \begin{pmatrix}-1\\3\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\5\end{pmatrix}$

$\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AG} = \begin{pmatrix}-1\\3\end{pmatrix} + \begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}1\\7\end{pmatrix}$

2 P(0, 2, –1) Q(2, 1, 1)

a $\overrightarrow{OP} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \overrightarrow{OQ} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

b $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

3 **a** $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

b $\overrightarrow{AD} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

c $\overrightarrow{AE} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

d $\overrightarrow{AG} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CG} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$

e $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix}$

f $\overrightarrow{BH} = \overrightarrow{BD} + \overrightarrow{DH} = \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$

4 P(–3, 1) Q(5, 7) R(–1, 5)

a $\overrightarrow{OP} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \overrightarrow{OQ} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \overrightarrow{OR} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

b M(1, 4) N(–2, 3)

c $\overrightarrow{QR} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix}$

$\overrightarrow{MN} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \therefore \overrightarrow{QR} = 2\,\overrightarrow{MN}$ (QED)

Exercise 11D

1 **a** $\mathbf{u} + (-\mathbf{u}) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} u_1 - u_1 \\ u_2 - u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$ (QED)

b $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right] = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$

$= \begin{pmatrix} u_1 + (v_1 + w_1) \\ u_2 + (v_2 + w_2) \end{pmatrix} = \begin{pmatrix} (u_1 + v_1) + w_1 \\ (u_2 + v_2) + w_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$= (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (QED)

c $\alpha(\beta\mathbf{u}) = \alpha\begin{pmatrix} \beta u_1 \\ \beta u_2 \end{pmatrix} = \begin{pmatrix} \alpha\beta u_1 \\ \alpha\beta u_2 \end{pmatrix} = \alpha\beta\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (\alpha\beta)\mathbf{u}$

$\alpha(\beta\mathbf{u}) = \begin{pmatrix} \alpha\beta u_1 \\ \alpha\beta u_2 \end{pmatrix} = \begin{pmatrix} \beta\alpha u_1 \\ \beta\alpha u_2 \end{pmatrix} = \beta\begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix} = \beta(\alpha\,\mathbf{u})$

(QED)

d $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\begin{pmatrix} u_1 + v_2 \\ u_2 + v_2 \end{pmatrix} = \begin{pmatrix} \alpha(u_1 + v_1) \\ \alpha(u_2 + v_2) \end{pmatrix} = \begin{pmatrix} \alpha u_1 + \alpha v_1 \\ \alpha u_2 + \alpha v_2 \end{pmatrix}$

$= \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix} + \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \end{pmatrix} = \alpha\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \alpha\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \alpha\mathbf{u} + \alpha\mathbf{v}$ (QED)

e $(\alpha + \beta)\mathbf{u} = (\alpha + \beta)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (\alpha + \beta)u_1 \\ (\alpha + \beta)u_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1 + \beta u_1 \\ \alpha u_2 + \beta u_2 \end{pmatrix}$

$= \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix} + \begin{pmatrix} \beta u_1 \\ \beta u_2 \end{pmatrix} = \alpha\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \beta\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha\mathbf{u} + \beta\mathbf{u}$ (QED)

f $0\mathbf{u} = 0\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0u_1 \\ 0u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$ (QED)

g $\alpha\mathbf{0} = \alpha\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha 0 \\ \alpha 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$ (QED)

2 **a** $2\begin{pmatrix} x \\ y \end{pmatrix} - 3\begin{pmatrix} y \\ x \end{pmatrix} = 5\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$2x - 3y = 5$

$2y - 3x = -10, \quad x = 4, \quad y = 1$

b $2\left(\begin{pmatrix} 2 \\ y \end{pmatrix} - \begin{pmatrix} x \\ 2 \end{pmatrix}\right) - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 0 \Rightarrow 2\begin{pmatrix} 2 - x \\ y - 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 0$

$4 - 2x - 1 = 0$

$2y - 4 - 3 = 0 \quad \therefore x = \frac{3}{2} \quad y = \frac{7}{2}$

3 **a** $\mathbf{u} + (\mathbf{v} + 2\mathbf{u}) = 3\mathbf{u} + \mathbf{v}$

b $(\mathbf{u} - \mathbf{v}) + 2(\mathbf{v} - 2\mathbf{u}) = \mathbf{u} - \mathbf{v} + 2\mathbf{v} - 4\mathbf{u} = -3\mathbf{u} + \mathbf{v}$

c $3\left(\frac{1}{6}(\mathbf{u} - \mathbf{v}) + \frac{1}{3}(\mathbf{v} - \mathbf{u})\right) = \frac{1}{2}(\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{u}) = -\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v}$

4 $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} \qquad \mathbf{b} = -\mathbf{i} - 2\mathbf{j}$

$\alpha\mathbf{a} + \beta\mathbf{b} = 3\mathbf{i} - \mathbf{j}$

$\therefore \alpha(2\mathbf{i} - 3\mathbf{j}) + \beta(-\mathbf{i} - 2\mathbf{j}) = 3\mathbf{i} - \mathbf{j}$

$\alpha\mathbf{a} + \beta\mathbf{b} = 3\mathbf{i} - \mathbf{j}$

$\therefore \alpha(2\mathbf{i} - 3\mathbf{j}) + \beta(-\mathbf{i} - 2\mathbf{j}) = 3\mathbf{i} - \mathbf{j}$

$2\alpha - \beta = 3, -3\alpha - 2\beta = -1 \therefore \alpha = 1 \quad \beta = -1$

$6\mathbf{i} - 2\mathbf{j} = 2\mathbf{a} - 2\mathbf{b}$

Exercise 11E

1 **a** $\mathbf{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \quad |\mathbf{v}| = \sqrt{26} \quad \frac{1}{\sqrt{26}}\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} \end{pmatrix}$

b $\mathbf{v} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} \quad |\mathbf{v}| = 13 \quad \frac{1}{13}\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} \\ \frac{12}{13} \end{pmatrix}$

c $\mathbf{v} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \quad |\mathbf{v}| = 3 \quad \frac{1}{3}\begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

d $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad |\mathbf{v}| = \sqrt{2} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

2 **a** $\mathbf{v}=\begin{pmatrix}-2\\1\end{pmatrix}$ $|\mathbf{v}|=\sqrt{5}$ $\pm\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}=\begin{pmatrix}-\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{pmatrix}$ or $\begin{pmatrix}\frac{2}{\sqrt{5}}\\-\frac{1}{\sqrt{5}}\end{pmatrix}$

b $\mathbf{v}=\begin{pmatrix}-5\\-2\end{pmatrix}$ $|\mathbf{v}|=\sqrt{29}$ $\pm\frac{1}{\sqrt{29}}\begin{pmatrix}-5\\-2\end{pmatrix}=\begin{pmatrix}-\frac{5}{\sqrt{29}}\\-\frac{2}{\sqrt{29}}\end{pmatrix}$ or $\begin{pmatrix}\frac{5}{\sqrt{29}}\\\frac{2}{\sqrt{29}}\end{pmatrix}$

c $\mathbf{v}=\begin{pmatrix}0\\-1\end{pmatrix}$ $\begin{pmatrix}0\\-1\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$

d $\mathbf{v}=\begin{pmatrix}-1\\-1\end{pmatrix}$ $|\mathbf{v}|=\sqrt{2}$ $\pm\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\-1\end{pmatrix}=\begin{pmatrix}-\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$ or $\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{pmatrix}$

3 $\mathbf{u}=2\mathbf{i}-3\mathbf{j}$ $|\mathbf{u}|=\sqrt{13}$ $\mathbf{v}=\frac{1}{\sqrt{13}}\mathbf{u}=\begin{pmatrix}\frac{2}{\sqrt{13}}\\\frac{-3}{\sqrt{13}}\end{pmatrix}$

4 **a** $|\mathbf{u}|=\sqrt{13}$ $\mathbf{v}=\pm\frac{2}{\sqrt{13}}$ $\mathbf{u}=\begin{pmatrix}\frac{4}{\sqrt{13}}\\\frac{6}{\sqrt{13}}\end{pmatrix}$ or $\begin{pmatrix}-\frac{4}{\sqrt{13}}\\-\frac{6}{\sqrt{13}}\end{pmatrix}$

b $|\mathbf{u}|=3$ $\mathbf{v}=\pm\frac{2}{3}$ $\mathbf{u}=\begin{pmatrix}-\frac{4}{3}\\\frac{2\sqrt{5}}{3}\end{pmatrix}$ or $\begin{pmatrix}\frac{4}{3}\\-\frac{2\sqrt{5}}{3}\end{pmatrix}$

c $|\mathbf{u}|=\sqrt{26}$ $\mathbf{v}=\pm\frac{13}{\sqrt{26}}\mathbf{u}=\pm\sqrt{\frac{13}{2}}\mathbf{u}=\begin{pmatrix}\frac{2}{\sqrt{13}}\\\frac{3}{\sqrt{13}}\end{pmatrix}$ or $\begin{pmatrix}\frac{-2}{\sqrt{13}}\\\frac{-3}{\sqrt{13}}\end{pmatrix}$

5 $\mathbf{u}=-4\mathbf{i}-6\mathbf{j}\ |\mathbf{u}|=\sqrt{52}=2\sqrt{13}$

$\mathbf{w}=\frac{1}{2}\mathbf{u}=-2\mathbf{i}-3\mathbf{j}$

6 $\mathbf{u}=\mathbf{i}-3\mathbf{j}$ $|\mathbf{u}|=\sqrt{10}$ $\mathbf{t}=\pm\frac{5}{\sqrt{10}}\mathbf{u}$

$\mathbf{t}=\frac{5}{\sqrt{10}}\mathbf{i}-\frac{15}{\sqrt{10}}\mathbf{j}$ or $\mathbf{t}=-\frac{5}{\sqrt{10}}\mathbf{i}+\frac{15}{\sqrt{10}}\mathbf{j}$

7 $\mathbf{u}=\frac{1}{\sqrt{v_1^2+v_2^2}}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\frac{1}{\sqrt{v_1^2+v_2^2}}\mathbf{v}$ $\therefore\mathbf{u}=k\mathbf{v}$ $(k>0)$

$\therefore$ **u** is in the same direction as **v** (QED)

$|\mathbf{u}|^2=\frac{v_1^2}{v_1^2+v_2^2}+\frac{v_2^2}{v_1^2+v_2^2}=\frac{v_1^2+v_2^2}{v_1^2+v_2^2}=1$ $\therefore|\mathbf{u}|=1$

$\therefore$ **u** has magnitude 1 (QED)

8 $\mathbf{u}=\pm\frac{\mathbf{m}}{|\mathbf{v}|}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\pm\begin{pmatrix}\frac{mv_1}{|\mathbf{v}|}\\\frac{mv_2}{|\mathbf{v}|}\end{pmatrix}$

$|\mathbf{u}|^2=\frac{m^2v_1^2}{|\mathbf{v}|^2}+\frac{m^2v_2^2}{|\mathbf{v}|^2}=\frac{m^2\left(v_1^2+v_2^2\right)}{v_1^2+v_2^2}=m^2$

$\therefore$ $|\mathbf{u}|$ = m (QED)

Exercise 11F

1 **a** A(2, 5) B(5, 6) C(4, 2) D(1, 1)

b $\overrightarrow{AB}=\begin{pmatrix}5\\6\end{pmatrix}-\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$

$\overrightarrow{AC}=\begin{pmatrix}4\\2\end{pmatrix}-\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}2\\-3\end{pmatrix}$

$\overrightarrow{AD}=\begin{pmatrix}1\\1\end{pmatrix}-\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}-1\\-4\end{pmatrix}$

c $\overrightarrow{BD}=\overrightarrow{BA}+\overrightarrow{AD}=-\begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}-1\\-4\end{pmatrix}=\begin{pmatrix}-4\\-5\end{pmatrix}$

2 A(2, 6) B(–2, 4)

a $\overrightarrow{AB}=\begin{pmatrix}-2\\4\end{pmatrix}-\begin{pmatrix}2\\6\end{pmatrix}=\begin{pmatrix}-4\\-2\end{pmatrix}$

b $AB=\sqrt{(-2-2)^2+(4-6)^2}=\sqrt{20}=2\sqrt{5}$

$\therefore|\overrightarrow{AB}|=2\sqrt{5}$

c M(0, 5) M is the midpoint of AB

d Let $P(x_1, y_1)$ $\overrightarrow{AP}=\begin{pmatrix}x_1-2\\y_1-6\end{pmatrix}$ $\overrightarrow{PB}=\begin{pmatrix}-2-x_1\\4-y_1\end{pmatrix}$

$\overrightarrow{AP}=2\overrightarrow{PB}\ \therefore x_1-2=2(-2-x_1), y_1-6=2(4-y_1)$

$x_1-2=-4-2x_1, y_1-6=8-2y_1$

$3x_1=-2$ $3y_1=14$

$x_1=\frac{-2}{3}$ $y_1=\frac{14}{3}$

$\therefore\ P\left(\frac{-2}{3},\frac{14}{3}\right)$

Let $Q(x_2, y_2)$, $\overrightarrow{AQ}=\begin{pmatrix}x_2-2\\y_2-6\end{pmatrix}$ $\overrightarrow{QB}=\begin{pmatrix}-2-x_2\\4-y_2\end{pmatrix}$

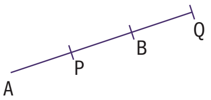
$\overrightarrow{AQ}=-2\overrightarrow{QB}$

$\therefore x_2-2=-2(-2-x_2), y_2-6=-2(4-y_2)$

$x_2-2=4+2x_2\ y_2-6=-8+2y_2$

$x_2=-6$ $y_2=2$

$\therefore$ Q(–6, 2)



$|\overrightarrow{AB}|<|\overrightarrow{PQ}|$

$\overrightarrow{PQ}$ has greater magnitude than $\overrightarrow{AB}$

3 P(4, –1) Q(6, –3) R(2, 1)

$\overrightarrow{PQ}=\begin{pmatrix}2\\-2\end{pmatrix}$ $\overrightarrow{PR}=\begin{pmatrix}-2\\2\end{pmatrix}$ $\overrightarrow{PR}=-1\overrightarrow{PQ}$

$\therefore$ $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are collinear

$\therefore$ P, Q, R are collinear (QED)

4 $A(a, a–1)$ $B(2, 2a)$ $C(0, 3a)$

$\overrightarrow{AB}=\begin{pmatrix}2-a\\a+1\end{pmatrix}$ $\overrightarrow{AC}=\begin{pmatrix}-a\\2a+1\end{pmatrix}$ $\overrightarrow{AC}=k\,\overrightarrow{AB}$

$\therefore\ -a=k(2-a)$

$2a+1=k(a+1)$ $\therefore\ \frac{-a}{2a+1}=\frac{2-a}{a+1}$

$-a^2-a=4a+2-2a^2-a$

$a^2-4a-2=0$

$a=\frac{4\pm\sqrt{16+8}}{2}, a=2\pm\sqrt{6}$

5 S(2, –3) U(–1, 2) N(1, –4)

$\overrightarrow{SU} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ $\overrightarrow{SN} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ $\overrightarrow{SN} \neq k\overrightarrow{SU}$

∴ S, U, N are not collinear

∴ they form a triangle (QED)

6 P(a, b) Q(c, d) R(e, f)

$\overrightarrow{PQ} = \begin{pmatrix} c-a \\ d-b \end{pmatrix}$ $\overrightarrow{PR} = \begin{pmatrix} e-a \\ f-b \end{pmatrix}$

$\frac{f-b}{d-b} = \frac{e-a}{c-a} \Rightarrow e - a = \frac{(f-b)(c-a)}{d-b}$

$\therefore \overrightarrow{PR} = \begin{pmatrix} \frac{(f-b)(c-a)}{d-b} \\ \frac{(f-b)(d-b)}{d-b} \end{pmatrix} = \frac{f-a}{d-b}\begin{pmatrix} c-a \\ d-b \end{pmatrix}$

$\therefore \overrightarrow{PR} = \frac{f-a}{d-b}\overrightarrow{PQ}$ $\therefore \overrightarrow{PR} = k\overrightarrow{PQ}$

∴ P, Q, R are collinear points

7 a $\overrightarrow{AB} = \begin{pmatrix} \sin 2x - \sin x \\ \cos 2x - (-1 + \cos x) \end{pmatrix}$

$= \begin{pmatrix} 2\sin x\cos x - \sin x \\ 2\cos^2 x - 1 + 1 - \cos x \end{pmatrix}$

$= \begin{pmatrix} \sin x(2\cos x - 1) \\ 2\cos^2 x - \cos x \end{pmatrix}$

$= \begin{pmatrix} \sin x(2\cos x - 1) \\ \cos x(2\cos x - 1) \end{pmatrix}$

$= 2\cos x - 1\begin{pmatrix} \sin x \\ \cos x \end{pmatrix}$

Therefore for any value of x, $\overrightarrow{AB}$ is collinear with $\begin{pmatrix} \sin x \\ \cos x \end{pmatrix}$

b $|\overrightarrow{AB}| = |2\cos x - 1|\sqrt{\sin^2 x + \cos^2 x}$

$= |2\cos x - 1|\sqrt{1}$

$= |2\cos x - 1|$

Exercise 11G

1 $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

a $\mathbf{u} + \mathbf{v} = -3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$

b $-3\mathbf{u} = 6\mathbf{i} - 9\mathbf{j} - 3\mathbf{k}$

c $4\mathbf{u} - 2\mathbf{v} = -8\mathbf{i} + 12\mathbf{j} + 4\mathbf{k} - (-2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k})$

$= -6\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}$

d $-2(\mathbf{u} - \mathbf{v}) = -2(-\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = 2\mathbf{i} - 2\mathbf{j} - 8\mathbf{k}$

2 $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

a $\mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}$ **b** $2\mathbf{a} - \mathbf{b} + \mathbf{c} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix}$

c $2(a - b) - 3c = 2\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -7 \end{pmatrix}$

d $\frac{1}{2}(\mathbf{a} - 3\mathbf{b}) = \frac{1}{2}\begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix}$

e $|\mathbf{a}| = \sqrt{14}$ **f** $|\mathbf{b}| = 3$

g $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} \therefore |\mathbf{a} + \mathbf{b}| = \sqrt{35}$

h $\mathbf{a} - \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \therefore |\mathbf{a} - \mathbf{b}| = \sqrt{11}$

3 A(0, 2, 1) B(–1, –1, –2) C(1, –3, 0)

a $\overrightarrow{AB} = \begin{pmatrix} -1 \\ -3 \\ -3 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} 1 \\ -5 \\ -1 \end{pmatrix}$

b $\overrightarrow{AB} - \overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$ $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

4 $v = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

a $\mathbf{u} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$ **b** $\pm\begin{pmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$ **c** $\begin{pmatrix} 0 \\ \sqrt{5} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$

5 A(4, –1, 3) C(0, –2, 5) D(5, 1, 6) G(1, –4, 6)

a $\overrightarrow{AC} = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$ $\overrightarrow{AD} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\overrightarrow{CG} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

b $\overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{AD} = \begin{pmatrix} -5 \\ -3 \\ -1 \end{pmatrix}$ $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}$

$\overrightarrow{AE} = \overrightarrow{CG} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix}$

$\overrightarrow{BF} = \overrightarrow{CG} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF} = \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix}$

$\overrightarrow{AH} = \overrightarrow{AD} - \overrightarrow{CG} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$ $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{AH} = \begin{pmatrix} 6 \\ -1 \\ 7 \end{pmatrix}$

Exercise 11H

1 $\overrightarrow{PQ} = \begin{pmatrix}1\\2\end{pmatrix}$ $r = \begin{pmatrix}1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\end{pmatrix}$

$x = 1 + \lambda,\ y = 3 + 2\lambda$

$x - 1 = \frac{y-3}{2}$

$2x - 2 = y - 3$

$y = 2x + 1$

2 $r = \begin{pmatrix}1\\-1\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$

$x = 1 + 2\lambda,\ y = -1 - \lambda,\ z = 1 + 3\lambda$

$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-1}{3}$

3 $\frac{x+1}{3} = \frac{2y}{3} = z - 1 \Rightarrow \frac{x+1}{3} = \frac{y}{\frac{3}{2}} = \frac{z-1}{1}$

$(-1, 0, 1)$ $\begin{pmatrix}3\\\frac{3}{2}\\1\end{pmatrix}$ or $\begin{pmatrix}6\\3\\2\end{pmatrix}$

4 **a** eg use $\lambda = 0, 1, -1$

$(1, 1, -1)$ $(0, 1, 2)$ $(2, 1, -4)$

(other solutions are possible)

b $x = 1 - \lambda$ $y = 1$ $z = -1 + 3\lambda$

At P, $y = 3 \neq 1$ $\therefore$ P does not lie on L (QED)

c $r = \begin{pmatrix}0\\3\\2\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\3\end{pmatrix}$

5 **a** eg use $k = 0, 1$ $(1, 0, 2)$ $(2, -1, 2)$

b direction of line $= \mathbf{i} - \mathbf{j}$

$\mathbf{u} = \pm\frac{4}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = \pm\left(2\sqrt{2}\mathbf{i} - 2\sqrt{2}\mathbf{j}\right) = \pm\begin{pmatrix}\frac{2}{\sqrt{2}}\\-\frac{2}{\sqrt{2}}\\0\end{pmatrix}$

Exercise 11I

1 $\mathbf{u} \cdot \mathbf{v} = 1.5 \times 4 \times \cos 30° = 3\sqrt{3}$

2 $\mathbf{u} \cdot (-\mathbf{v}) = |\mathbf{u}|\ |-\mathbf{v}| \cos(\pi - \theta)$

$= |\mathbf{u}|\ |\mathbf{v}| \cos(\pi - \theta)$

$= -|\mathbf{u}|\ |\mathbf{v}| \cos\theta$

$= -(\mathbf{u} \cdot \mathbf{v})$

$(-\mathbf{u}) \cdot \mathbf{v} = |-\mathbf{u}|\ |-\mathbf{v}| \cos(\pi - \theta)$

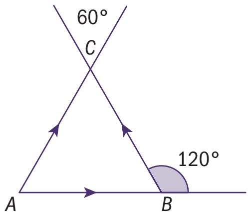
$= |\mathbf{u}|\ |\mathbf{v}| \cos(\pi - \theta)$

$= -|\mathbf{u}|\ |\mathbf{v}| \cos\theta$

$= -(\mathbf{u} \cdot \mathbf{v})$

$\therefore \mathbf{u} \cdot (-\mathbf{v}) = -(\mathbf{u} \cdot \mathbf{v}) = (-\mathbf{u}) \cdot \mathbf{v}$ (QED)

3



Let the length of the sides be x

$\overrightarrow{AB} \cdot \overrightarrow{BC} = x^2 \cos 120° = -\frac{1}{2}x^2$

$\overrightarrow{BC} \cdot \overrightarrow{AC} = x^2 \cos 60° = \frac{1}{2}x^2$

$\therefore \overrightarrow{AB} \cdot \overrightarrow{BC} + \overrightarrow{BC} \cdot \overrightarrow{AC} = \frac{-1}{2}x^2 + \frac{1}{2}x^2 = 0$

4 **a** $\overrightarrow{AB} \cdot \overrightarrow{AC} = xy\cos\alpha = x^2$

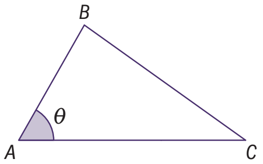
b $\overrightarrow{CA} \cdot \overrightarrow{CB}$

$= y\sqrt{y^2 - x^2}\cos\left(\frac{\pi}{2} - \alpha\right) = y\sqrt{y^2 - x^2}\sin\alpha = y^2\sin^2\alpha$

c $\overrightarrow{AC} \cdot \overrightarrow{CB}$

$= y\sqrt{y^2 - x^2}\cos\left(\frac{\pi}{2} + \alpha\right) = -y\sqrt{y^2 - x^2}\sin\alpha = -y^2\sin^2\alpha$

5



Area = 4

$\therefore \frac{1}{2} \times 2 \times 5\sin\theta = 4$

$\therefore \sin\theta = \frac{4}{5}$

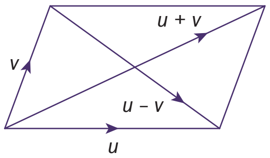
$\therefore \cos\theta = \pm\frac{3}{5}$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 2 \times 5\cos\theta = \pm 6$

6 Angle between **u** and **u** = 0

$\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|\ |\mathbf{u}| \cos 0 = |\mathbf{u}|^2$ (QED)

7



if $|\mathbf{u} + \mathbf{v}| = |\mathbf{u} - \mathbf{v}|$ the parallelogram is a rectangle and the angle between **u** and **v** is $\frac{\pi}{2}$

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\ |\mathbf{v}| \cos\frac{\pi}{2} = 0$ (QED)

8 diagonal AG $= \sqrt{7^2 + 2^2 + 3^2} = \sqrt{62}$

a $OB = OC = \frac{\sqrt{62}}{2}$ $\cos B\hat{O}C = \frac{\left(\frac{\sqrt{62}}{2}\right)^2 + \left(\frac{\sqrt{62}}{2}\right)^2 - 2^2}{2 \times \frac{\sqrt{62}}{2} \times \frac{\sqrt{62}}{2}} = \frac{27}{31}$

$\overrightarrow{OB} \cdot \overrightarrow{OC} = \left(\frac{\sqrt{62}}{2}\right)\left(\frac{\sqrt{62}}{2}\right)\left(\frac{27}{31}\right) = \frac{27}{2}$

b $OA = OE = \frac{\sqrt{62}}{2}$ $\cos A\hat{O}E = \frac{\left(\frac{\sqrt{62}}{2}\right)^2 + \left(\frac{\sqrt{62}}{2}\right)^2 - 3^2}{2 \times \frac{\sqrt{62}}{2} \times \frac{\sqrt{62}}{2}} = \frac{22}{31}$

$\overrightarrow{OA} \cdot \overrightarrow{OE} = \left(\frac{\sqrt{62}}{2}\right)\left(\frac{\sqrt{62}}{2}\right)\left(\frac{22}{31}\right) = 11$

Exercise 11J

1 **a** $\mathbf{u} \cdot \mathbf{v} = -12 + 24 = 12$

b $\mathbf{u} \cdot \mathbf{v} = -1 - 3 + 10 = 6$

2 A(–1, 3, 2) B(–1, 1, 2) C(1, –1, 1)

$\overrightarrow{AB} = \begin{pmatrix}0\\-2\\4\end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix}2\\-2\\-1\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}2\\-4\\3\end{pmatrix}$

$\overrightarrow{AB} \cdot \overrightarrow{BC} = 0 + 4 - 4 = 0$

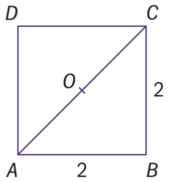
$\overrightarrow{AC} \cdot \overrightarrow{BC} = 4 + 8 - 3 = 9$

3 **a** 0(0, 0, 0) A(0, 0, 1) B(1, 0, 1) C(1, 0, 0)
D(0, 1, 0) E(0, 1, 1) F(1, 1, 1) G(1, 1, 0)

b $\overrightarrow{OF}\cdot\overrightarrow{OG} = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} = 2$

$\overrightarrow{AF} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ $\overrightarrow{BG} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$ $\overrightarrow{AF}\cdot\overrightarrow{BG} = 1$

4 **a**



$AC = 2\sqrt{2}$

$\therefore OA = OB = OC = OD = \sqrt{2}$

Volume $= \frac{1}{3} \times 4 \times OE = \frac{8}{3} \therefore OE = 2$

$A(0, \sqrt{2}, 0)$ $B(\sqrt{2}, 0, 0)$ $C(0, \sqrt{2}, 0)$
$D(-\sqrt{2}, 0, 0)$ $E(0, 0, 2)$

b $\overrightarrow{EA} = \begin{pmatrix}0\\-\sqrt{2}\\-2\end{pmatrix}$ $|\overrightarrow{EA}| = \sqrt{6}$

$\overrightarrow{EB} = \begin{pmatrix}\sqrt{2}\\0\\-2\end{pmatrix}$ $\overrightarrow{EA}\cdot\overrightarrow{EB} = 4$

c $\cos A\hat{E}B = \frac{6+6-4}{2\sqrt{6}\sqrt{6}} = \frac{8}{12}$

$A\hat{E}B = 48.2°$

Exercise 11K

1 $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$

$\mathbf{u}\cdot\mathbf{v} = 2 - 6 = -4$ $|\mathbf{u}| = \sqrt{13}$ $|\mathbf{v}| = \sqrt{5}$

$\cos\theta = \frac{-4}{\sqrt{13}\sqrt{5}}$ $\therefore \theta = 120°$

2 $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

$\mathbf{u}\cdot\mathbf{v} = 2 + 2 + 1 = 5$ $|\mathbf{u}| = \sqrt{6}$ $|\mathbf{v}| = \sqrt{6}$

$\cos\theta = \frac{5}{6} \therefore \theta = 33.6°$

3 A(−1, 1, 1) B(1, −1, 2) C(2, 3, −1)

a $\overrightarrow{AB} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}3\\2\\-2\end{pmatrix}$

$\overrightarrow{AB}\cdot\overrightarrow{AC} = 6 - 4 - 2 = 0$

$\therefore \cos\theta = 0 \therefore \theta = 90°$

b $\overrightarrow{BC} = \begin{pmatrix}1\\4\\-3\end{pmatrix}$ $\overrightarrow{BC}\cdot\overrightarrow{AC} = 3 + 8 + 6 = 17$

$|\overrightarrow{BC}| = \sqrt{26}$ $|\overrightarrow{AC}| = \sqrt{17}$

$\cos\theta = \frac{17}{\sqrt{26}\sqrt{17}}$ $\therefore \theta = 36.0°$

4 $\mathbf{u}\cdot\mathbf{v} > 0 \therefore a(a-2) + 3(a-4) > 0$

$a^2 + a - 12 > 0$

When $(a-3)(a+4) > 0$

$a = 3$ or -4 $\therefore a < -4$ or $a > 3$

5 $\mathbf{u} = \sin 3\alpha\mathbf{i} - \cos 3\alpha\mathbf{j} + 2\mathbf{k}$

$\mathbf{v} = \cos\alpha\mathbf{i} - \sin\alpha\mathbf{j} - 2\mathbf{k}$

a $\mathbf{u}\cdot\mathbf{v} = \sin 3\alpha\cos\mu + \cos 3\alpha\sin\alpha - 4$

$= \sin 4\alpha - 4$

$|\mathbf{u}|^2 = \sin^2 3\alpha + \cos^2 3\alpha + 4 = 5$ $\therefore |\mathbf{u}| = \sqrt{5}$

$|\mathbf{v}|^2 = \cos^2\alpha + \sin^2\alpha + 4 = 5$ $\therefore |\mathbf{u}| = \sqrt{5}$

$\therefore \cos\theta = \frac{\sin 4\alpha - 4}{5}$

b $\cos 150° = \frac{\sin 4\alpha - 4}{5}$

$-\frac{\sqrt{3}}{2} = \frac{\sin 4\alpha - 4}{5}$

$\sin 4\alpha = -0.3301$

$4\alpha = 3.478, 5.947, 9.761, 12.230, 16.044, 18.513, 22.328, 27.796$

$\alpha = 0.870, 1.49, 2.44, 3.06, 4.01, 4.63, 5.58, 6.20$

c $\sin 4\alpha < 4 \therefore \sin 4\alpha - 4 < 0$

$\therefore \cos\theta < 0 \therefore \theta$ is obtuse (QED)

6 Let $\mathbf{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ $\mathbf{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ $\mathbf{c} = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$ $\mathbf{d} = \begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$

$\mathbf{a} + \mathbf{b} = \begin{pmatrix}a_1+b_1\\a_2+b_2\\a_3+b_3\end{pmatrix}$ $\mathbf{c} + \mathbf{d} = \begin{pmatrix}c_1+d_1\\c_2+d_2\\c_3+d_3\end{pmatrix}$

$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{c}+\mathbf{d}) = (a_1+b_1)(c_1+d_1) + (a_2+b_2)(c_2+d_2) + (a_3+b_3)(c_3+d_3)$

$= a_1c_1 + a_1d_1 + b_1c_1 + b_1d_1 + a_2c_2 + a_2d_2 + b_2c_2 + b_2d_2 + a_3c_3 + a_3d_3 + b_3c_3 + b_3d_3$

$= (a_1c_1 + a_2c_2 + a_3c_3) + (a_1d_1 + a_2d_2 + a_3d_3) + (b_1c_1 + b_2c_2 + b_3c_3) + (b_1d_1 + b_2d_2 + b_3d_3)$

$= \mathbf{a}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d}$ (QED)

Exercise 11L

1 $\mathbf{u} = \mathbf{i} - \mathbf{j} - \mathbf{k}$ $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ $\mathbf{w} = \mathbf{i} - \mathbf{k}$

a $\mathbf{u}\times\mathbf{v} = -3\mathbf{i} - 2\mathbf{j} - \mathbf{k} = \begin{pmatrix}-3\\-2\\-1\end{pmatrix}$

b $\mathbf{v}\times\mathbf{w} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = \begin{pmatrix}2\\2\\2\end{pmatrix}$

c $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = -4\mathbf{j} + 4\mathbf{k} = \begin{pmatrix}0\\-4\\4\end{pmatrix}$

d $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} = \begin{pmatrix}2\\-4\\2\end{pmatrix}$

e $\mathbf{u}\times\mathbf{w} = \mathbf{i} + \mathbf{k} = \begin{pmatrix}1\\0\\1\end{pmatrix}$

f $(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\mathbf{w}) = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = \begin{pmatrix}-2\\2\\2\end{pmatrix}$

g $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{w}) = (2\mathbf{i} - 3\mathbf{j})\times(-\mathbf{j}) = -2\mathbf{k} = \begin{pmatrix}0\\0\\-2\end{pmatrix}$

2 $\mathbf{u} = \mathbf{i} - \mathbf{j} - \mathbf{k} \quad \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \quad \mathbf{w} = \mathbf{i} - \mathbf{k}$

From qn. 1, $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = -4\mathbf{j} + 4\mathbf{k}$

$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

$\therefore\ \mathbf{u}\times(\mathbf{v}\times\mathbf{w}) \neq (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$

$\therefore$ not associative (QED)

3 $\mathbf{w}\cdot\mathbf{u} = 0$ and $\mathbf{w}\cdot\mathbf{v} = 0$

$\therefore\ w_1 u_1 + w_2 u_2 + w_3 u_3 = 0$ and

$w_1 v_1 + w_2 v_2 + w_3 v_3 = 0$

$\mathbf{u}\times\mathbf{v} = (u_2 v_3 - u_3 v_2)\,\mathbf{i} + (u_3 v_1 - u_1 v_3)\,\mathbf{j} + (u_1 v_2 - u_2 v_1)\,\mathbf{k}$

$$\begin{aligned}\mathbf{w}\times(\mathbf{u}\times\mathbf{v}) &= [w_2(u_1 v_2 - u_2 v_1) - w_3(u_3 v_1 - u_1 v_3)]\,\mathbf{i}\\ &\quad + [w_3(u_2 v_3 - u_3 v_2) - w_1(u_1 v_2 - u_2 v_1)]\,\mathbf{j}\\ &\quad + [w_1(u_3 v_1 - u_1 v_3) - w_2(u_2 v_3 - u_3 v_2)]\,\mathbf{k}\\ &= [u_1(w_2 v_2 + w_3 v_3) - v_1(w_2 u_2 + w_2 u_3)]\,\mathbf{i}\\ &\quad + [u_2(w_3 v_3 + w_1 v_1) - v_2(w_3 u_3 + w_1 u_1)]\,\mathbf{j}\\ &\quad + [u_3(w_1 v_1 + w_2 v_2) - v_3(w_1 u_1 + w_2 u_2)]\,\mathbf{k}\\ &= [u_1(-w_1 v_1) - v_1(-w_1 u_1)]\,\mathbf{i}\\ &\quad + [u_2(-w_2 v_2) - v_2(-w_2 u_2)]\,\mathbf{j}\\ &\quad + [u_3(-w_3 v_3) - v_3(-w_3 u_3)]\,\mathbf{k} = 0\end{aligned}$$

$\mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = 0\ \therefore\ \mathbf{w}$ and $\mathbf{u}\times\mathbf{v}$ are collinear

4 A(–1, 3, 4) B(5, 7, 5) C(3, 9, 6)

a $\overrightarrow{AB} = \overrightarrow{AC} = \begin{pmatrix}-2\\2\\1\end{pmatrix} \quad \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \begin{pmatrix}-3\\5\\5\end{pmatrix}$

$\therefore$ D (–3, 5, 5)

b $\overrightarrow{AB} = \begin{pmatrix}6\\4\\1\end{pmatrix} \quad \overrightarrow{BC} = \begin{pmatrix}-2\\2\\1\end{pmatrix}$

$\overrightarrow{AB}\times\overrightarrow{BC} = \begin{pmatrix}2\\-8\\20\end{pmatrix}$

c area $= \left|\overrightarrow{AB}\times\overrightarrow{BC}\right| = \sqrt{4+64+400}$

$= \sqrt{468} = 2\sqrt{117}$

$= 6\sqrt{13}$

5 **a** A(1, 0, 2) B(2, 3, 3) C(–3, –1, 2) E(2, 1, 4)

$\overrightarrow{AD} = \overrightarrow{BC} = \begin{pmatrix}-5\\-4\\-1\end{pmatrix} \quad \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \begin{pmatrix}-4\\-4\\1\end{pmatrix}$

D (–4, –4, 1)

b Volume $= (\overrightarrow{AB}\times\overrightarrow{AD})\cdot\overrightarrow{AE}$

$\overrightarrow{AB}\times\overrightarrow{AD} = \begin{pmatrix}1\\3\\1\end{pmatrix}\times\begin{pmatrix}-5\\-4\\-1\end{pmatrix} = \begin{pmatrix}1\\-4\\11\end{pmatrix}$

Volume $= \begin{pmatrix}1\\-4\\11\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix} = 1 - 4 + 22$

= 19 cu. units

6 **a** $\mathbf{u}\times\mathbf{v} = (u_2 v_3 - u_3 v_2)\,\mathbf{i} + (u_3 v_1 - u_1 v_3)\,\mathbf{j} + (u_1 v_2 - u_2 v_1)\,\mathbf{k}$

$= -(v_2 u_3 - v_3 u_2)\,\mathbf{i} - (v_3 u_1 - v_1 u_3)\,\mathbf{j} - (v_1 u_2 - v_2 u_1)\,\mathbf{k}$

$\mathbf{v}\times\mathbf{u} = (v_2 u_3 - v_3 u_2)\mathbf{i} + (v_3 u_1 - v_1 u_3)\,\mathbf{j} + (v_1 u_2 - v_2 u_1)\,\mathbf{k}$

$\therefore\ \mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$ (QED)

b $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}\times\begin{pmatrix}v_1+w_1\\v_2+w_2\\v_3+w_3\end{pmatrix}$

$= [u_2(v_3 + w_3) - u_3(v_2 + w_2)]\,\mathbf{i} + [u_3(v_1 + w_1) - u_1(v_3 + w_3)]\,\mathbf{j} + [u_1(v_2 + w_2) - u_2(v_1 + w_1)]\,\mathbf{k}$

$[(u_2 v_3 - u_3 v_2) + (u_2 w_3 - u_3 w_2)]\,\mathbf{i}$

$+ [(u_3 v_1 - u_1 v_3) + (u_3 w_1 - u_1 w_3)]\,\mathbf{j}$

$+ [(u_1 v_2 - u_2 v_1) + (u_1 w_2 - u_1 w_3)]\,\mathbf{k}$

$= (u_2 v_3 - u_3 v_2)\,\mathbf{i} + (u_3 v_1 - u_1 v_3)\,\mathbf{j}$

$+ (u_1 v_2 - u_2 v_1)\,\mathbf{k}$

$+ (u_2 w_3 - u_3 w_2)\,\mathbf{i} + (u_3 w_1 - u_1 w_3)\,\mathbf{j}$

$+ (u_1 w_2 - u_2 w_1)\,\mathbf{k}$

$\therefore\ \mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$ (QED)

c $\mathbf{u}\times\mathbf{v}\cdot\mathbf{u} = \begin{pmatrix}u_2v_3 - u_3v_2\\u_3v_1 - u_1v_3\\u_1v_2 - u_2v_1\end{pmatrix}\bullet\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$

$= u_1 u_2 v_3 - u_1 u_3 v_2 + u_2 u_3 v_1 - u_1 u_2 v_3 + u_1 u_3 v_2 - u_2 u_3 v_1 = 0$

$\therefore\ \mathbf{u}\times\mathbf{v}\cdot\mathbf{u} = 0$ (QED)

d $(\lambda\,\mathbf{u})\times\mathbf{v} = \begin{pmatrix}\lambda u_1\\\lambda u_2\\\lambda u_3\end{pmatrix}\times\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}\lambda u_2v_3 - \lambda u_3v_2\\\lambda u_3v_1 - \lambda u_1v_3\\\lambda u_1v_2 - \lambda u_2v_1\end{pmatrix}$

$= \lambda\begin{pmatrix}u_2v_3 - u_3v_2\\u_3v_1 - u_1v_3\\u_1v_2 - u_2v_1\end{pmatrix} = \lambda\,(\mathbf{u}\times\mathbf{v})$

$\therefore\ (\lambda\mathbf{u})\times\mathbf{v} = \lambda\,(\mathbf{u}\times\mathbf{v})$ (QED)

Exercise 11M

1 A(–3, 1, 1) B(1, 2, 0) C(1, 1, –2)

a $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} \overrightarrow{AC} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$

$$r = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

b $x = -3 + 4\alpha + 4\beta$ (1)

$y = 1 + \alpha$ (2)

$z = 1 - \alpha - 3\beta$ (3)

c From (2) $\alpha = y - 1$

From (1) $4\beta = x + 3 - 4(y - 1)$

$\beta = \frac{x - 4y + 7}{4}$

sub in (3): $z = 1 - y + 1 - \frac{3}{4}(x - 4y + 7)$

$4z = 8 - 4y - 3x + 12y - 21$

$3x - 8y + 4z = -13$ (other forms are possible)

2 P(1, 0, 1) $\mathbf{n} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$$

$-x + 3y - z = -2$

$x - 3y + z = 2$

eg (0, 0, 2) $(0, -\frac{2}{3}, 0)$ (2, 0, 0)

3 **a** (–3, 4, 0) $\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

$x - 2y - z = -11$

b P(–1, 1, 1) $x - 1 = \frac{y - 1}{2} = z$

line passes through A(1, 1, 0) and has direction $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$\overrightarrow{AP} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix}$$

$-2x + 3y - 4z = 1$

$2x - 3y + 4z = -1$

c $1 - x = y - 1 = 2z$ and $x = 2 - t$

$\frac{x - 1}{-1} = \frac{y - 1}{1} = \frac{z}{\frac{1}{2}}$ $y = 1 + 2t$

$z = t$

(2, 1, 0) lies in the plane

$\begin{pmatrix} -1 \\ 1 \\ \frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ are vectors in the plane

$$\mathbf{n} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$y - 2z = 1$

4 **a** $y = 0$ **b** $z = 0$ **c** $x = 0$

5 **a** $\frac{1}{3} \times 4 \times h = 6 \therefore h = \frac{9}{2}$

0(0, 0, 0) A(2, 0, 0) B(2, 2, 0) C(0, 2, 0)
$V(0, 0, \frac{9}{2})$

b $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ $\overrightarrow{AV} = \begin{pmatrix} -2 \\ 0 \\ \frac{9}{2} \end{pmatrix}$

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ \frac{9}{2} \end{pmatrix}$$

c $\overrightarrow{CB} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ $\overrightarrow{CV} = \begin{pmatrix} 0 \\ -2 \\ \frac{9}{2} \end{pmatrix}$

$$n = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -4 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ -18 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -9 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -9 \\ -4 \end{pmatrix}$$

$-9y - 4z = -18$

$9y + 4z = 18$

d $\overrightarrow{VB} = \begin{pmatrix} 2 \\ 2 \\ -\frac{9}{2} \end{pmatrix}$ $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ -9 \end{pmatrix}$

e $r = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 9 \end{pmatrix}$

$x = 2 - 4\lambda,\ y = 0,\ z = 9\lambda$

$\lambda = \frac{x-2}{-4}\ \lambda = \frac{z}{9}$

$\frac{x-4}{-4} = \frac{z}{9} \Rightarrow 9x - 36 = -4z$

$z = \frac{36-9x}{4},\ y = 0$

Exercise 11N

1 $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad y = \sqrt{3}x - 3$

Method 1 $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$

$\cos\theta = \left|\frac{1-\sqrt{3}}{\sqrt{2}\sqrt{4}}\right| \ \therefore\ \theta = 75°$

Method 2 $\mathrm{m}_1 = -1 \quad \mathrm{m}_2 = \sqrt{3}$

$\tan\mu = -1\ \tan\beta = \sqrt{3}$

$\therefore\ \theta = -45°\ \therefore\ \beta = 60°$

$\theta = 1\beta - \theta 1 = 105°$

$\therefore$ acute angle = 75°

2 $\frac{x-1}{2} = \frac{y}{3} = \frac{z}{1} \quad 2x = y = 3z$

$\frac{x}{\frac{1}{2}} = \frac{y}{1} = \frac{z}{\frac{1}{3}}$

$\frac{x}{3} = \frac{y}{6} = \frac{z}{2}$

$\mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} \quad \mathbf{u}\cdot\mathbf{v} = 6 + 18 + 2 = 26$

$|\mathbf{u}| = \sqrt{14} \quad |\mathbf{v}| = 7$

$\cos\theta = \frac{26}{7\sqrt{14}}\ \therefore\ \theta = 6.93°$

3 $\mathbf{r} = (2 - \theta)\,\mathbf{i} + (- + 2\theta)\,\mathbf{j} + (1 - \theta)\,\mathbf{k}$

$\mathbf{r} = (2 + \beta)\,\mathbf{j} + (3 + \beta)\,\mathbf{k}$

$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{u}\cdot\mathbf{v} = 1$

$\cos\theta = \frac{1}{\sqrt{6}\sqrt{2}}\ \therefore\ \theta = 73.2°$

4 a $\mathbf{u} = \begin{pmatrix} 1 \\ m_1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ m_2 \end{pmatrix} \quad \mathbf{u}\ \mathbf{v} = 1 + m_1\, m_2$

$|\mathbf{u}| = \sqrt{1+m_1^2} \quad |\mathbf{v}| = \sqrt{1+m_2^2}$

$\cos\theta = \frac{|\mathbf{u}\cdot\mathbf{v}|}{|\mathbf{u}|\ |\mathbf{v}|}$

$\therefore\ \cos\theta = \frac{|1+m_1m_2|}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}$

b $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{m_1 - m_2}{1+m_1m_2}$

c $\sec^2(\alpha - \beta) = 1 + \tan^2(\alpha - \beta)$

$= 1 + \frac{(m_1 - m_2)^2}{(1+m_1m_2)^2}$

$= \frac{(1+m_1m_2)^2 + (m_1 - m_2)^2}{(1+m_1m_2)^2}$

$= \frac{1+2m_1m_2+m_1^2m_2^2+m_1^2-2m_1m_2+m_2^2}{(1+m_1m_2)}$

$= \frac{1+m_1^2+m_2^2+m_1^2m_2^2}{(1+m_1m_2)^2}$

$= \frac{(1+m_1^2)(1+m_2^2)}{(1+m_1m_2)}$

$\therefore\ \cos(\alpha - \beta) = \frac{|1+m_1m_2|}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}$ (> 0 since $\alpha > \beta$)

d $\cos\theta = \cos(\alpha - \beta)$ since $\theta = \alpha - \beta$

Exercise 11O

1 $\mathbf{r} = (1 - 2\lambda)\,\mathbf{i} + (1 - \lambda)\,\mathbf{j} + (-2 + \lambda)\,\mathbf{k} \quad \mathbf{u} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

$2x - y + z = 5 \quad \mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\sin\theta = \frac{|\mathbf{u}\cdot\mathbf{n}|}{|\mathbf{u}|\ |\mathbf{n}|} = \frac{|-2|}{\sqrt{6}\sqrt{6}} \quad \theta = 195°$

2 $\frac{x-1}{3} = 2y = 3 - 2z \Rightarrow \frac{x-1}{3} = \frac{y}{\frac{1}{2}} = \frac{z - \frac{3}{2}}{\frac{-1}{2}}$

$\Rightarrow \frac{x-1}{6} = \frac{y}{1} = \frac{z - \frac{3}{2}}{-1} \Rightarrow \mathbf{u} = \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}$

$\mathbf{r} = (2 - 2\theta - 3\beta)\,\mathbf{i} + (1 - \theta + \beta)\,\mathbf{j} + (-2\theta + \beta)\,\mathbf{k}$

$\mathbf{n} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}$

$\mathbf{u}\cdot\mathbf{n} = 6 + 8 + 5 = 19$

$\sin\theta = \frac{19}{\sqrt{38}\sqrt{90}} \quad \therefore\ \theta = 19.0°$

3 $x - y + 3z = 1 \quad \mathbf{m} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

$\mathbf{r} = (4 - 2\theta + 2\beta)\,\mathbf{i} + (1 - 3\beta)\,\mathbf{j} + (2 - \theta - \beta)\,\mathbf{k}$

$\mathbf{n} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 6 \end{pmatrix} \quad \mathbf{m}\cdot\mathbf{n} = -3 + 4 + 18 = 19$

$\cos\theta = \frac{19}{\sqrt{11}\sqrt{61}} \quad \theta = 42.8°$

4 A(1, 0, 1) B(–1, 1, 0) C(2, 3, –1) D(–1, –1, –1)

a $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$

$$\mathbf{n} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix}$$

$x - 5y - 7z = -6$

b $\overrightarrow{AD} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}$ $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$x = 1 + 2\lambda$ $y = \lambda$ $z = 1 + 2\lambda$

$\frac{x-1}{2} = y = \frac{z-1}{2}$

c $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $\mathbf{n} = \begin{pmatrix} 1 \\ -5 \\ -7 \end{pmatrix}$ $\mathbf{u}\cdot\mathbf{n} = 2 - 5 - 14 = -17$

$\sin\theta = \frac{17}{3\sqrt{75}}$ $\theta = 40.9°$

5 $\frac{x}{2} = \mathrm{k}y = \mathrm{k} - z$ $(2\mathrm{k} - 1)\,x - \mathrm{k}y + z = 5 + \mathrm{k}$

$\frac{x}{-2k} = \frac{y}{-1} = \frac{z-k}{k}$ $u = \begin{pmatrix} -2k \\ -1 \\ k \end{pmatrix}$ $n = \begin{pmatrix} 2k-1 \\ -k \\ 1 \end{pmatrix}$

If the line and plane are parallel, $\mathbf{u}\cdot\mathbf{n} = 0$

so $-\,2k(2k - 1) + k + k = 0$

$\Rightarrow -4k^2 + 4k = 0$

$\Rightarrow -4k(k - 1) = 0 \Rightarrow k = 0$ or 1

Exercise 11P

1 $\mathbf{r} = \mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

P$(1 + \lambda, -2\lambda, 1 + \lambda)$ $x + y + 2z = 4$

$1 + \lambda - 2\lambda + 2(1 + \lambda) = 4$

$\lambda = 1$ P(2, –2, 2)

2 $\frac{x-1}{5} = \frac{y}{2} = \frac{z}{3} = \lambda$ $x - 1 + 5\lambda, y = 2\lambda, z = 3\lambda$

$-x - y + 3z = 5$

$-1 - 5\lambda - 2\lambda + 9\lambda = 5$

$\therefore 2\lambda = 6 \therefore \lambda = 3$ (16, 6, 9)

3 $x = 3k, y = 2 - 2k, z = 1 - k$

$\mathbf{r} = (4 - 2\theta + \beta)\,\mathbf{i} + (1 - \beta)\,\mathbf{j} + (2 - \theta - 2\beta)\,\mathbf{k}$

a $x = 4 - 2\theta + \beta$

$y = 1 - \beta$

$z = 2 - \theta - 2\beta$

b $3k = 4 - 2\theta + \beta \Rightarrow 3k + 2\theta - \beta = 4$

$2 - 2k = 1 - \beta \Rightarrow -2k + \beta = -1$

$1 - k = 2 - \theta - 2\beta \Rightarrow \Rightarrow k + \theta + 2\beta = 1$

$k = 0.6, \theta = 1.2, \beta = 0.2$ (1.8, 0.8, 0.4)

4 $\mathbf{r} = (1 + \lambda)\,\mathbf{i} + (1 + 2\lambda)\,\mathbf{j} + (1 + \lambda)\,\mathbf{k}$, $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$3x - y - z = 2$ $n = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

$\mathbf{u}\cdot\mathbf{n} = 3 - 2 - 1 = 0 \therefore$ **u** and **n** are perpendicular

∴ the line parallel to the plane (QED)

(0, 0, –2) lies in the plane

$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ lies in the plane

$\mathbf{r} = \lambda\mathbf{i} + 2\lambda\,\mathbf{j} + (-2 + \lambda)\,\mathbf{k}$ (other answers are possible)

5 P(1, 2, 3) $2x + y - 5z = 1$ $\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$

line through p perpendicular to the plane:

$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$

point of intersection, I$(1 + 2\lambda, 2 + \lambda, 3 - 5\lambda)$

$2(1 + 2\lambda) + 2 + \lambda - 5(3 - 5\lambda) = 1$

$2 + 4\lambda + 2 + \lambda - 15 + 25\lambda = 1$

$30\lambda = 12 \Rightarrow \lambda = 0.4$

I(1.8, 2.4, 1)

$\mathrm{PI} = \sqrt{(1.8-1)^2 + (2.4-2)^2 + (1-3)^2}$

$= \sqrt{4.8} = \frac{1}{10}\sqrt{480}$

distance $= \frac{2}{5}\sqrt{30}$ or 2.19

Exercise 11Q

1 **a** $\frac{x}{2} = y - 1 = z$ (1) $x = \frac{y+4}{3} = 3 - z$ (2)

$x = 2y - 2$ $3x = y + 4$

$6y - 6 = y + 4$

$5y = 10 \Rightarrow y = 2, x = 2$

sub in (1) sub in (2)

$\frac{2}{2} = 2 - 1 = z$ $2 = \frac{2+4}{3} = 3 - z$

$z = 1$ $z = 1$

∴ intersect at (2, 2, 1)

b $\mathbf{r} = (5 + 2\lambda)\,\mathbf{i} + (4 + \lambda)\,\mathbf{j} + (5 - 3\lambda)\,\mathbf{k}$, $x = y = z + 1$

P$(5 + 2\lambda, 4 + \lambda, 5 - 3\lambda)$

$5 + 2\lambda = 4 + \lambda$ and $4 + \lambda = 5 - 3\lambda + 1$

$\lambda = -1$ $4\lambda = 2$

$\lambda = \frac{1}{2}$

Equations are inconsistent ∴ no point of intersection

2 $x - 3y + 8 = 2$ (1)

$-x + y - 2z = 1$ (2)

(1) + (2) $-2y - z = 3 \Rightarrow z = -2y - 3$

3(2) + (1) $-2x - 5z = 5 \Rightarrow z = \frac{-2x-5}{5}$

$\frac{-2x-5}{5} = -2y - 3 = z$

$\frac{2x+5}{5} = 2y + 3 = \frac{z}{-1}$

$\frac{x+2.5}{5} = \frac{y+1.5}{1} = \frac{z}{-2}$

3 **a** $3x - y + z = 3 \quad \mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

$x = 3 - (2 - \text{k})\,\lambda \; y = (2\text{k} - 1) + \lambda$

$z = -1 + \text{k}\lambda \quad \mathbf{u} = \begin{pmatrix} k-2 \\ 1 \\ k \end{pmatrix}$

n and **u** are collinear $\therefore k = -1$

b $3x - y + z = 3 \; x = 3 - 3\lambda,\; y = -3 + \lambda,$

$z = -1 - \lambda$

$3(3 - 3\lambda) - (-3 + \lambda) + (-1 - \lambda) = 3$

$9 - 9\lambda + 3 - \lambda - 1 - \lambda = 3$

$-11\lambda = -8$

$\lambda = \frac{8}{11} \left(\frac{9}{11}, \frac{-25}{11}, \frac{-19}{11}\right)$

4 **a** L_1: $y = 2x + 2,\; z = 3 - x$

$\frac{x}{1} = \frac{y-2}{2} = \frac{z-3}{-1}$ direction, $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

L_2: $\frac{x-1}{3} = \frac{y-1}{6} = \frac{1-z}{3} \Rightarrow \frac{x-1}{3} = \frac{y-1}{6} = \frac{z-1}{-3}$

direction, $\mathbf{v} = \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix}$

$\mathbf{v} = 3\mathbf{u} \therefore \text{L}_1$ and L_2 are parallel (QED)

b A(0, 2, 3) and B(1, 1, 1) are points in the plane

$\overrightarrow{\text{AB}} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$

$5x - y + 3z = 7$

5 $x + y + 3z = 1 \quad x = 4 - y$ and $z = -1$

$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \frac{x}{1} = \frac{y-4}{-1} \quad \mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\mathbf{u} \cdot \mathbf{n} = 1 - 1 + 0 = 0$

$\therefore$ **u** and **n** are perpendicular $\therefore$ the line is parallel to the plane.

(0, 4, –1) lies on the line

$x + y + 3z = 0 + 4 + 3(-1) = 1$

$\therefore$ (0, 4, –1) also lies in the plane

$\therefore$ the plane contains the line (QED)

Exercise 11R

1 $5x + y + 2z = 3$ (1) (1) – (3) $x - y = -2$ (4)

$x + y + z = 3$ (2) (1) – 2(2) $3x - y = -3$ (5)

$4x + 2y + 2z = 5$ (3) (5) – (4) $2x = -1$

$\therefore x = \frac{-1}{2}\; y = \frac{3}{2}$

From (2) $z = 3 - x - y = 2 \left(\frac{-1}{2}, \frac{3}{2}, 2\right)$

3 $x + y + z = 1$ (1) (1) – (2) $2y = -2$

$x - y + z = 3$ (2) $y = -1$

$3x + y + 3z = 1$ (3)

sub in (1) and (3) $x + z = 2$ (4)

$3x + 3z = 2$ (5)

(4) and (5) are inconsistent $\therefore$ no common point (QED)

4 **a** $x + y + z = 0$ (1) (2) – (1) $ax - x = 0$

$ax + y + z = 0$ (2) $x(a - 1) = 0$

$x + by + cz = z = 0$ (3) $x = 0$ or $a = 1$

if $x = 0$, $y + z = 0$ either $y = z = 0$

$by + cz = 0$ or b = c

If $x = 0$, either point of intersection (0, 0, 0) or intersect in a straight line.

if a = 1 equations (1) and (2) are the same

$x + y + z = 0$

$x + \text{b}y + \text{c}z = 0$

$x = y = z = 0$ or b = c = 1 all 3 planes the same or *b* and *c* not both = 1 and intersect in a straight line.

$\therefore$ if $a = 1$, either a point of intersection (0, 0, 0) or intersect in a plane or interest in a line.

$\therefore$ the planes always have at least one point in common.

b For a straight line, $b = c$ but a, b, c not all equal to 1 or $a = 1$ and b and c both equal to 1

$x + 2y - 2z = 5 \; \pi_1$

$3x - 6y + 3z = 2 \; \pi_2$

$x - 2y + z = 7 \; \pi_3$

π_2 and π_3 are parallel but not coincident, π intersects each of the other 2 please in a straight line but the 3 planes have no point in common.

6 **a** $x + y + z = 2$ (1) (1) + (2) $3x + 2z = 1$ (4)
$2x - y + z = -1$ (2) (1) + (3) $4x - 2z = 6$ (5)
$3x - y - 3z = 4$ (3) (4) + (5) $7x = 7$
$x = 1, z = -1, y = 2$
(1, 2, –1)

b $x + y + z = 2$ (1) (1) + (2) $3x + 2z = 1$
$2x - y + z = -1$ (2) (1) + (3) $4x + (k + 1)z = 6$
$3x - y + kz = 4$ (3)
For no common point, $\frac{4}{3} = \frac{k+1}{2}$ ($\neq 6$)

$8 = 3k + 3 \therefore k = \frac{5}{3}$

7 Verify by diagrams

Exercise 11S

1 A $x = 3 - t$ B $x = 4 - 3t$
$y = 2t - 4$ $y = 3 - 2t$

a A(3, –4) B(4, 3)

b $V_A = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $V_B = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

c $\cos\theta = \frac{V_A \cdot V_B}{|V_A|\,|V_B|} = \frac{3-4}{\sqrt{5}\sqrt{13}} = \frac{-1}{\sqrt{65}}$ $\theta = 97.1°$

d $\overrightarrow{AB} = \begin{pmatrix} 4-3t \\ 3-2t \end{pmatrix} - \begin{pmatrix} 3-t \\ 2t-4 \end{pmatrix} = \begin{pmatrix} 1-2t \\ 7-4t \end{pmatrix}$

$AB = \sqrt{(1-2t)^2 + (7-4t)^2}$
This is a minimum when $t = 1.5$ hours

2 **a** $\overrightarrow{OP} = (5 + 10t)\,\mathbf{i} + (20 - 20t)\,\mathbf{j} + (30t - 10)\,\mathbf{k}$,
$t > 0$ $t = 0$, P(5, 20, –10)

b Cartesian equations:

$\frac{x-5}{10} = \frac{y-20}{-20} = \frac{z+10}{30}$ (×10)

$\frac{x-5}{1} = \frac{y-20}{-2} = \frac{z+10}{3}$

c **i** $x + y + z = 55$
$5 + 10t + 20 - 20t + 30t - 10 = 55$
$20t = 40$
$t = 2$

ii (25, –20, 50)

iii $P_0P_2 = \begin{pmatrix} 20 \\ -40 \\ 60 \end{pmatrix}$ distance

$= \sqrt{20^2 + (-40)^2 + 60^2} = 20\sqrt{14}$

d $\overrightarrow{OQ} = \begin{pmatrix} 2t^2 \\ 1-2t \\ 1+t^2 \end{pmatrix}$ $t > 0$

i $\overrightarrow{PQ} = \begin{pmatrix} 2t^2 - 5 - 10t \\ 1 - 2t - 20 + 20t \\ 1 + t^2 - 30t + 10 \end{pmatrix} = \begin{pmatrix} 2t^2 - 10t - 5 \\ 18t - 19 \\ t^2 - 30t + 11 \end{pmatrix}$

$PQ = \sqrt{(2t^2 - 10t - 5)^2 + (18t - 19)^2 + (t^2 - 30t + 11)^2}$

This is a minimum when t = 0.49598817
= 0.496 (3sf)

ii P(9.96, 10.1, 4.88)
Q(0.492, 0.00802, 1.25)

e **i** $a = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $b = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ $c = \begin{pmatrix} 8 \\ -3 \\ 5 \end{pmatrix}$

$a - b = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$ $b - c = \begin{pmatrix} -6 \\ 2 \\ -3 \end{pmatrix}$

$a - b \neq k(b - c) \therefore a - b$ and $b - c$ are non collinear (QED)

ii Q is not moving in a straight line

3 **a** B(1, 1, 0) C(0, 1, 0) $\therefore \overrightarrow{OP} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

C(0, 1, 0) G(0, 1, 1) $\therefore \overrightarrow{OQ} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix}$

D(0, 0, 1) G(0, 1, 1) $\therefore \overrightarrow{OR} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$

b $\overrightarrow{PQ} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$ $\overrightarrow{PR} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

$n = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

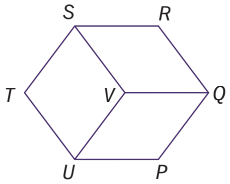
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$x + y + z = \frac{3}{2}$ or $2x + 2y + 2z = 3$

c E(1, 0, 1) d(0, 0, 1) $\therefore \overrightarrow{OS} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$

A(1, 0, 0) E(1, 0, 1) $\therefore \overrightarrow{OT} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix}$

A(1, 0, 0) B(1, 1, 0) $\therefore \overrightarrow{OU} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$



S, T, U lie in the plane PQR

$\overrightarrow{PQ}\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$ $\overrightarrow{QR} = \begin{pmatrix} 0 \\ \frac{-1}{2} \\ \frac{1}{2} \end{pmatrix}$ $\overrightarrow{RS} = \begin{pmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 0 \end{pmatrix}$ $\overrightarrow{ST} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{-1}{2} \end{pmatrix}$

$\overrightarrow{TU} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{-1}{2} \end{pmatrix}$ $\overrightarrow{UP} = \begin{pmatrix} \frac{-1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$

all sides are of length $\frac{1}{\sqrt{2}}$

cos P = cos Q = cos R = cos S = cos T

$= \cos U = \dfrac{\frac{-1}{4}}{\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}} = \frac{-1}{2}$

$\therefore$ all angles are 120° $\therefore$ PQRSTU is a regular hexagon

Area parallelogram PQVU $= \left|\overrightarrow{PQ} \times \overrightarrow{PU}\right|$

$\begin{pmatrix} \frac{-1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$ $\therefore$ area PQVU $= \frac{\sqrt{3}}{4}$

$\therefore$ area hexagon $= \frac{3\sqrt{3}}{4}$

d $\overrightarrow{OF} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = n$ $\therefore \overrightarrow{OF}$ is perpendicular to the plane PQR (QED)

e General point on OF is $(\lambda, \lambda, \lambda)$

$2x + 2y + 2z = 3 \quad 2\lambda + 2\lambda + 2\lambda = 3$

$\therefore \lambda = \frac{3}{1} = \frac{1}{2} \quad I\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

f $\overrightarrow{IF} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ $\therefore$ distance IF $= \frac{\sqrt{3}}{2}$

Review exercise

1 $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\mathbf{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ **a** $3\mathbf{u} - 2\mathbf{v} = \begin{pmatrix} 9 \\ 12 \end{pmatrix} - \begin{pmatrix} -2 \\ 10 \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \end{pmatrix}$

b $|\mathbf{u}| = 5 \ |\mathbf{v}| = \sqrt{26} \ \mathbf{u} + \mathbf{v} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$

$\therefore |\mathbf{u} + \mathbf{v}| = \sqrt{85}$

2 $\mathbf{u} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \quad \mathbf{a} = \mathbf{i} + \mathbf{j} \quad \mathbf{b} = \mathbf{i} + \mathbf{k}$

$\mathbf{c} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

$\mathbf{u} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$

$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \gamma\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

$\therefore \alpha + \beta + 2r = 3$ (1) (1) – (2) $\beta + 3\gamma = 4$ (4)

$\alpha - \gamma = -1$ (2) $\beta - \gamma = 1$ (3)

$\beta - \gamma = 1$ (3) (4) – (3) $4\gamma = 3$

$\gamma = \frac{3}{4}$

$\gamma = \frac{3}{4}, \beta = \frac{7}{4}, \alpha = -\frac{1}{4} \therefore \mathbf{u} = -\frac{1}{4}\mathbf{a} + \frac{7}{4}\mathbf{b} + \frac{3}{4}\mathbf{c}$

3 **a** $a = \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$ $|a| = 5\sqrt{2}$ $\therefore$ unit vector $= \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-4}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} \end{pmatrix}$

b $\pm \frac{5}{5\sqrt{2}}\begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} = \pm \begin{matrix} \frac{-5}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{matrix}$

4 $\mathbf{u} = \cos\alpha\cos\beta\,\mathbf{i} + \sin^2\alpha\cos^2\beta\,\mathbf{j} + \sin^2\beta\,\mathbf{k}$

$|\mathbf{u}|^2 = \cos^2\alpha\cos^2\beta + \sin^2\alpha\cos^2\beta + \sin^2\beta$

$= \cos^2\beta(\cos^2\alpha + \sin^2\alpha) + \sin^2\beta$

$= \cos^2\beta + \sin^2\beta$

$= 1$

$\therefore |\mathbf{u}| = 1 \therefore \mathbf{u}$ is a unit vector

5 $\mathbf{u} = \mathbf{i} + \tan\alpha\,\mathbf{j} \quad \mathbf{v} = \tan\beta\,\mathbf{i} + \mathbf{j}$

$\cos r = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\ |\mathbf{v}|} = \frac{\tan\beta + \tan\propto}{\sqrt{1+\tan^2\propto}\sqrt{1+\tan^2\beta}}$

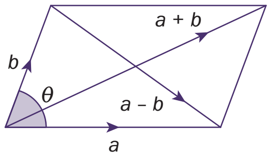
$= \frac{\tan\beta + \tan\propto}{\sec\propto\sec\beta}$

$= (\tan\beta + \tan\alpha)\cos\alpha\cos\beta$

$= \sin\beta\cos\alpha + \sin\alpha\cos\beta$

$$\cos \gamma = \sin(\alpha + \beta)$$
$$= \cos\left(\frac{\pi}{2} - (\alpha + \beta)\right)$$
$$\therefore\ \gamma = \frac{\pi}{2} - (\alpha + \beta)\ \therefore\ \alpha + \beta + \gamma = \frac{\pi}{2}$$

6



a $|\mathbf{a}| = 1\ |\mathbf{b}| = 1$

$|\mathbf{a} - \mathbf{b}|^2 = 1^2 + 1^2 - 2(1)(1)\cos\theta$

$|\mathbf{a} - \mathbf{b}|^2 = 2 - 2\cos\theta$

$|\mathbf{a} + \mathbf{b}|^2 = 1^2 + 1^2 - 2(1)(1)\cos(\pi - \theta)$

$= 2\ \theta\ 2\cos(\pi - \theta)$

$|\mathbf{a} + \mathbf{b}|^2 = 2 + 2\cos\theta$

b $|\mathbf{a} + \mathbf{b}| = 2|\mathbf{a} - \mathbf{b}| \Rightarrow |\mathbf{a} + \mathbf{b}|^2 = 4\ |\mathbf{a} + \mathbf{b}|^2$

$2 + 2\cos\theta = 4(2 - 2\cos\theta)$

$2 + 2\cos\theta = 8 - 8\cos\theta$

$10\cos\theta = 6\ \therefore\ \cos\theta = \frac{3}{5}\ (0 \le \theta \le \frac{\pi}{2})$

$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}\ \therefore\ \sin\theta = \frac{4}{5}$

7 **a** $\mathbf{r} = (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) + \alpha(2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) + \beta(\mathbf{j} - 2\mathbf{k})$

$x = 2 + 2\alpha,\ y = 3 - 3\alpha + \beta,\ z = 4 + 2\alpha - 2\beta$

b $\mathbf{r} = (i - 2k) + \alpha(-2i + k) + \beta(-j)$

$x = 1 - 2\alpha,\ y = -\beta,\ z = -2 + \alpha$

8 $5 + 3\lambda = -2 + 4\mu$ (1)

$1 - 2\lambda = 2 + \mu$ (2)

From (2) $\mu = -2\lambda - 1$

Sub in (1) $5 + 3\lambda = -2 - 8\lambda - 4$

$11\lambda = -11\ \therefore\ \lambda = -1\ \alpha = 1$ P(2, 3)

Review exercise

1 $\mathbf{u} = \begin{pmatrix}1\\1\\3\end{pmatrix}$ $\mathbf{v} = \begin{pmatrix}2\\0\\1\end{pmatrix}$ $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\ |\mathbf{v}|}$

$\cos\theta = \frac{2+3}{\sqrt{11}\sqrt{5}} = \frac{5}{\sqrt{55}}$

$\theta = 48°$ (nearest degree)

2 **a** $\overrightarrow{AB} = \begin{pmatrix}-2\\0\\2\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}-1\\0\\-1\end{pmatrix}$

$$\mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \alpha\begin{pmatrix}-2\\0\\2\end{pmatrix} + \beta\begin{pmatrix}-1\\0\\-1\end{pmatrix}$$

$x = 1 - 2\alpha - \beta,\ y = 1,\ z = 2\alpha - \beta$

$$\mathbf{n} = \begin{pmatrix}-2\\0\\2\end{pmatrix} \times \begin{pmatrix}-1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\-4\\0\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}0\\-4\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-4\\0\end{pmatrix}$$

$-4y = -4$ or $y = 1$

b $\overrightarrow{AB} = \begin{pmatrix}-1\\-2\\1\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}1\\0\\-2\end{pmatrix}$

$$\mathbf{r} = \begin{pmatrix}-1\\1\\1\end{pmatrix} + \alpha\begin{pmatrix}-1\\-2\\1\end{pmatrix} + \beta\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

$x = -1 - \alpha + \beta,\ y = 1 - 2\alpha,\ z = 1 + \alpha - 2\beta$

$$\mathbf{n} = \begin{pmatrix}-1\\-2\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\-2\end{pmatrix} = \begin{pmatrix}4\\-1\\2\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}4\\-1\\2\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}4\\-1\\2\end{pmatrix}$$

$4x - y + 2z = -3$

3 **a** $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix} = \begin{pmatrix}2\\-3\\4\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}$

or $\mathbf{r}\cdot(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = -9$

b $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\mathbf{r}\,(\mathbf{i} - \mathbf{j}) = 1$

4 **a** $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\4\end{pmatrix} = \begin{pmatrix}2\\-3\\4\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\4\end{pmatrix}$

$2x - 3y + 4z = 29$

b $\overrightarrow{AB} = \begin{pmatrix}-6\\0\\-3\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}-3\\6\\0\end{pmatrix}$ $\mathbf{n} = \begin{pmatrix}2\\0\\1\end{pmatrix} \times \begin{pmatrix}-1\\2\\0\end{pmatrix} = \begin{pmatrix}-2\\-1\\4\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}-2\\-1\\4\end{pmatrix} = \begin{pmatrix}6\\0\\0\end{pmatrix}\cdot\begin{pmatrix}-2\\-1\\4\end{pmatrix}$$

$-2x - y + 4z = -12$

$2x + y - 4z = 12$

c $\overrightarrow{AB} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ $\mathbf{n} = \begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}-6\\3\\0\end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$2x - y = 4$

d $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$ $\mathbf{n} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

$5y + 2z = -11$

e $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$x = 3$

f $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$x - y - z = 0$

g $x + y + 5z = 0$ (1)

$2x + 3y + 12z = 0$ (2)

$2 - 2(1)\ y + 2z = 0$

Let $z = \lambda$, $y = -2\lambda$, $x = -y - 5z = 2\lambda - 5\lambda$

$x = -3\lambda$

Line of intersection is $\mathbf{r} = \lambda \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$

$\mathbf{n} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0$$

$-x + 2y + z = 0$

or $x - 2y - z = 0$

5 $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$\cos\theta = \dfrac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}|\ |\mathbf{v}|} = \dfrac{|2 - 1 - 2|}{\sqrt{6}\sqrt{6}} = \dfrac{1}{6}$

$\theta = 80.4°$

6 $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

$\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 5\mathbf{k} + \alpha(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

$1 + \lambda = 1 + 2\alpha$ (1)

$1 + 2\lambda = 4 + \alpha$ (2)

$1 + 3\lambda = 5 + 2\alpha$ (3)

From (1) $\lambda = 2\alpha$

Sub in (2) $1 + 4\mu = 4 + \mu$

$\therefore 3\alpha = 3 \therefore \alpha = 1, \lambda = 2$

Check in (3) $1 + 3(2) = 5 + 2(1)$

$7 = 7$

$\therefore$ L_1 and L_2 are concurrent

point of intersection is (3, 5, 7)

$\mathbf{r} = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k} + \alpha(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \beta(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

7 $x + y + z = 3$ (1) (2) – (1) $x - 3z = -3$ (4)

$2x + y - 2z = 0$ (2) (3) + 2(1) $5x + 7z = 29$ (5)

$3x - 2y + 5z = 23$ (3)

(5) – 5(4) $22z = 44$

$\therefore z = 2, x = 3$

$y = 2z - 2x = -2$

$\therefore x = 3, y = -2, z = 2$

(3, –2, 2) is the point of intersection of the 3 planes represented by the equations.

8 a $3x + y + z = 1$ (1) (1) – (2) $2x + 2z = -3$ (4)

$x + y - z = 4$ (2) (1) – (3)

$x + (1 - b)z = 1 - a$ (5)

$2x + y + bz = a$ (3)

(4) –2(5) $2z - 2(1 - b)\ z = -3 - 2(1 - a)$

$2bz = -5 + 2a$

$z = \dfrac{2a - 5}{2b}$

$2x = -3 - 2z = -3 - \dfrac{(2a - 5)}{b} = \dfrac{-3b - 2a + 5}{b}$

$x = \dfrac{5 - 2a - 3b}{2b}$

$y = 4 - x + z = \dfrac{8b - (5 - 2a - 3b) + 2a - 5}{2b}$

$y = \dfrac{-10 + 4a + 11b}{2b}$

$x = \dfrac{5 - 2a - 3b}{2b}, y = \dfrac{4a + 11b - 10}{2b}, z = \dfrac{2a - 5}{2b}$

b $b = 0$ equations (4) and (5) become

$2x + 2z = -3$

$x + z = 1 - a$

For a non-unique solution, $1 - a = -\dfrac{3}{2}$

$\therefore a = \dfrac{5}{2}$

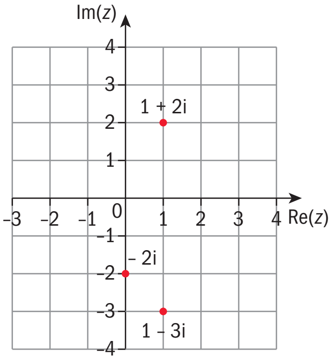
$a = \dfrac{5}{2}$, $b = 0$ the planes represented by the 3 equations intersect in a line.

12 Multiple perspectives in mathematics

Answers

Skills check

1



2 $z = 5 - 4i$

a $z^* = 5 + 4i \; -z = -5 + 4i$

b $\frac{1}{z} = \frac{1}{5-4i} \times \frac{5+4i}{5+4i}$

$= \frac{5}{41} + \frac{4}{41}i$

3 $z = -3 + 4i$

a $\text{Re}(z) = -3$

b $\text{Im}(z) = 4$

c $|z| = \sqrt{(-3)^2 + 4^2} = 5$

4 $z_1 = 1 - 2i \quad z_2 = 3 + i$

a $z_1 + 2z_2 = 1 - 2i + 6 + 2i = 7$

b $3z_1 z_2 + z_1^2 = 3(1 - 2i)(3 + i) + (1 - 2i)^2$

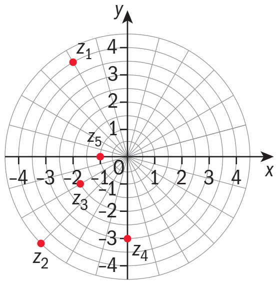
$= 3(3 + i - 6i + 2) + (1 - 4i - 4)$

$= 12 - 19i$

c $\frac{z_1}{z_2} = \frac{1-2i}{3+i} \times \frac{3-i}{3-i} = \frac{3-i-6i-2}{10} = \frac{1}{10} - \frac{7}{10}i$

Exercise 12A

1



$z_1 = z_2 \Rightarrow |z_1| = |z_2|$

$r^3 = r^2 + 2r$

$r(r^2 - r - 2) = 0$

$r(r - 2)(r + 1) = 0$

$r = 0 \text{ or } 2 \; (r \geq 0)$

$\arg(z_1) - \arg(z_2) = 2k\pi \quad k \in \mathbb{Z}$

$4\theta - (\theta + \frac{\pi}{2}) = 2k\pi$

$3\theta = \frac{\pi}{2} + 2k\pi$

$\theta = \frac{\pi}{6} + \frac{2k\pi}{3} \; k \in \mathbb{Z}$

Complex numbers are given by $z_1 = z_2 = 0$

or $z_1 = z_2 = 8 \text{ cis } \frac{2\pi}{8}$

$z_1 = z_2 = 8 \text{ cis } \frac{4\pi}{3}$

$z_1 = z_2 = 8 \text{ cis } 2\pi$

i.e. $0, -4 + 4i\sqrt{3}, -4 - 4i\sqrt{3}, 8$

3 $|a + ai| = 2 \Rightarrow \sqrt{a^2 + a^2} = 2$

$2a^2 = 4$

$a^2 = 2$

$a = \pm\sqrt{2}$

$\arg(a + ai) = \theta \; \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$

2 distinct complex numbers, $\sqrt{2} + \sqrt{2}i$ and $-\sqrt{2} - \sqrt{2}i$

4 **a** eg. $z_1 = 3 + 3i \quad z_2 = 4 + 4i$

$z_1 + z_2 = 7 + 7i \quad z_1 - z_2 = -1 - i$

$\arg(z_1) = \frac{\pi}{4} \quad \arg(z_2) = \frac{\pi}{4}$

$\arg(z_1 + z_2) = \frac{\pi}{4} \neq \arg(z_1) + \arg(z_2)$

$\arg(z_1 + z_2) = \frac{5\pi}{4} \neq \arg(z_1) - \arg(z_2)$

$\therefore \arg(z_1 \pm z_2) \neq \arg(z_1) \pm \arg(z_2)$ (QED)

b eg. $z_1 = 3 + 2i \quad z_2 = -1 + 4i$

$z_1 + z_2 = 2 + 6i \quad z_1 - z_2 = 4 - 2i$

$|z_1| = \sqrt{13} \quad |z_2| = \sqrt{17}$

$|z_1 + z_2| = \sqrt{40} \neq |z_1| + |z_2|$

$|z_1 - z_2| = \sqrt{20} \neq |z_1| + |z_2|$

$\therefore |z_1 \pm z_2| \neq |z_1| \pm |z_2|$

Exercise 12B

1 **a** $x = 6\cos 45° = 3\sqrt{2} \quad y = 6\sin 45° = 3\sqrt{2}$

$z_1 = 3\sqrt{2} + 3\sqrt{2}i$

b $x = 10\cos 135° = -5\sqrt{2} \quad y = 10\sin 135° = 5\sqrt{2}$

$z_2 = -5\sqrt{2} + 5\sqrt{2}i$

c $z_3 = 4 \text{ cis } \frac{5\pi}{3} = 4(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})$

$= 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2 - 2\sqrt{3}i$

d $z_4 = 5 \text{ cis } \frac{7\pi}{6} = 5(\cos \frac{7\pi}{6} + \text{i}\sin \frac{7\pi}{6})$

$= 5\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\text{i}\right) = -\frac{5\sqrt{3}}{2} - \frac{5}{2}\text{i}$

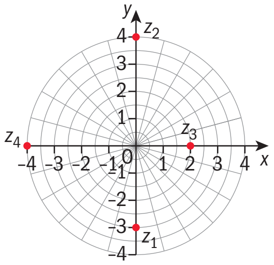
2 **a** $z_1 = -1 - \text{i}$ $\quad r = \sqrt{2}\,\theta = \frac{5\pi}{4}$ $\quad z_1 = \sqrt{2} \text{ cis } \frac{5\pi}{4}$

b $z_2 = 2\sqrt{3} + 2\text{i}$ $\quad r = 4\theta = \frac{\pi}{6}$ $\quad z_2 = 4 \text{ cis } \frac{\pi}{6}$

c $z_3 = 4 - 4\text{i}$ $\quad r = 4\sqrt{2}\,\theta = \frac{7\pi}{4}$ $\quad z_3 = 4\sqrt{2} \text{ cis } \frac{7\pi}{4}$

d $z_4 = -5 + 5\text{i}$ $\quad r = 5\sqrt{2}\,\theta = \frac{3\pi}{4}$ $\quad z_4 = 5\sqrt{2} \text{ cis } \frac{3\pi}{4}$

3



a $z_1 = -3\text{i} = 3 \text{ cis } \frac{3\pi}{2}$

b $z_2 = 4\text{i} = 4 \text{ cis } \frac{\pi}{2}$

c $z_3 = 2 = 2 \text{ cis } 0$

d $z_4 = -4 = 4 \text{ cis } \pi$

4 $z = 4 \text{ cis } 40°$

a $z^* = 4 \text{ cis } (-40°)$ or $4 \text{ cis } 320°$

b $-z = 4 \text{ cis } 220°$

c $-z^* = 4 \text{ cis } 140°$

d $3z^* = 12 \text{ cis } (-40°)$ or $12 \text{ cis } 320°$

e $-4z^* = 16 \text{ cis } 140°$

5 $z_1 = -2 - 2\sqrt{3}\text{i}$ $\quad z_2 = 3\sqrt{3} + 3\text{i}$

a $z_3 = (-2 - 2\sqrt{3}\text{i})\,(3\sqrt{3} + 3\text{i})$

$= -6\sqrt{3} - 6\text{i} - 18\text{i} + 6\sqrt{3}$

$= -24\text{i}$

b $|z_1| = \sqrt{4 + 12} = 4$ $\quad \arg(z_1) = \frac{4\pi}{3}$ $\quad z_1 = 4 \text{ cis } \frac{4\pi}{3}$

$|z_2| = \sqrt{27 + 9} = 6$ $\quad \arg(z_2) = \frac{\pi}{6}$ $\quad z_2 = 6 \text{ cis } \frac{\pi}{6}$

$|z_3| = 24$ $\quad \arg(z_3) = \frac{3\pi}{2}$ $\quad z_3 = 24 \text{ cis } \left(\frac{3\pi}{2}\right)$

c $|z_3| = |z_1|\,|z_2|,$ $\quad \arg(z_3) = \arg(z_1) + \arg(z_2)$

Exercise 12C

1 **a** $z_1 z_2 = 5 \text{ cis } 135°$

b $z_1 z_2 = \frac{2}{7} \text{ cis } \frac{31\pi}{24}$

2 $z_1 = \text{cis } \frac{5\pi}{6}$ $\quad z_2 = 1 - \text{i}$

a $z_1 = \cos \frac{5\pi}{6} + \text{i} \sin \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}$

b $z_2 = \sqrt{2} \text{ cis } \frac{7\pi}{4}$

c $z_1 z_2 = \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}\right)(1 - \text{i}) = -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\text{i} - \frac{1}{2}\text{i} + \frac{1}{2}$

$= -\frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}+1}{2}\text{i}$

$z_1 z_2 = \sqrt{2} \text{ cis } \left(\frac{5\pi}{6} + \frac{7\pi}{4}\right) = \sqrt{2} \text{ cis } \frac{31\pi}{12} = \sqrt{2} \text{ cis } \frac{7\pi}{12}$

$\sqrt{2} \sin \frac{7\pi}{12} = \frac{\sqrt{3}+1}{2} \therefore \sin \frac{7\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$

$\sqrt{2} \cos \frac{7\pi}{12} = -\frac{\sqrt{3}+1}{2} \therefore \cos \frac{7\pi}{12} = -\frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$

$\tan \frac{7\pi}{12} = -\frac{\sqrt{3}+1}{\sqrt{3}+1} + \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{4+2\sqrt{3}}{-2} = -2 - \sqrt{3}$

3 $z_1 = 2 \text{ cis } \frac{\pi}{6}$ $\quad z_2 = r \text{ cis } \theta$ $\quad r > 0,\ 0 < \theta < 2\pi$

a $z_1 z_2 = 2\,r \text{ cis } \left(\frac{\pi}{6} + \theta\right)$

$z_1 z_2$ real $\Rightarrow \sin\left(\frac{\pi}{6} + \theta\right) = 0$

$\frac{\pi}{6} + \theta = \pi$ or 2π $\quad \theta = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$

$z_1 z_2 > 5 \Rightarrow \theta = \frac{11\pi}{6}$ and $2r > 5 \Rightarrow r > \frac{5}{2}$

$r > \frac{5}{2}$ and $\theta = \frac{11\pi}{6}$

b $z_1 z_2$ imaginary $\Rightarrow \cos\left(\frac{\pi}{6} + \theta\right) = 0$

$\frac{\pi}{6} + \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\theta = \frac{\pi}{3}$ or $\frac{4\pi}{3}$

$|z_1 z_2| < 1 \Rightarrow 2r < 1 \Rightarrow r < \frac{1}{2}$

$0 < r < \frac{1}{2}$ and $\theta = \frac{\pi}{3}$ or $\frac{4\pi}{3}$

($r \neq 0$ otherwise $z_1 z_2$ is real)

4 $z = -3 + \sqrt{3}\text{i} = 2\sqrt{3} \text{ cis } \frac{5\pi}{6}$

$-z = z\sqrt{3} \text{ cis } \frac{11\pi}{6},\ z^* = 2\sqrt{3} \text{ cis } \left(-\frac{5\pi}{6}\right)$ or $2\sqrt{3} \text{ cis } \frac{7\pi}{6}$

$-z^* = 2\sqrt{3} \text{ cis } \frac{\pi}{6}$

5 $z = \sin \alpha + \cos \alpha\, \text{i}$ $\quad w = \sin 2\alpha - \cos 2\alpha\, \text{i}$

a $|z| = 1$ $\quad \tan \theta = \cot \alpha \therefore$ $\quad \theta = \frac{\pi}{2} - \alpha,$

$z = \text{cis } \left(\frac{\pi}{2} - \alpha\right)$

$|w| = 1$ $\quad \tan \theta = -\cot 2\alpha$

$\theta = -\left(\frac{\pi}{2} - 2\alpha\right)$ or $2\pi - \left(\frac{\pi}{2} - 2\alpha\right)$

$\theta = \frac{3\pi}{2} + 2\alpha \therefore w = \text{cis } \left(\frac{3\pi}{2} + 2\alpha\right)$

b $|zw| = 1\ \arg(zw) = \frac{\pi}{2} - \alpha + \frac{3\pi}{2} + 2\alpha$

$= 2\pi + \alpha$ or α

$zw = \text{cis } \alpha$

Exercise 12D

1 **a** $\frac{z_1}{z_2} = 2 \text{ cis } 45°$

b $\frac{z_1}{z_2} = \frac{7}{8} \text{ cis } \frac{25\pi}{24}$

2 $z_1 = 2 \text{ cis } \frac{11\pi}{6}$ $\quad z_2 = 2 - 2\text{i}$

a $|z_2| = 2\sqrt{2}\ \arg(2 - 2\text{i}) = \frac{7\pi}{4}\ z_2 = 2\sqrt{2} \text{ cis } \frac{7\pi}{4}$

b $z_2^* = 2\sqrt{2} \text{ cis } \frac{\pi}{4}$

c $z_1 z_2 = 4\sqrt{2}\ \text{cis}\ \frac{43\pi}{12} = 4\sqrt{2}\ \text{cis}\ \frac{19\pi}{12}$

d $\frac{z_1}{z_2} = \frac{1}{\sqrt{2}}\ \text{cis}\ \frac{\pi}{12} = \frac{\sqrt{2}}{2}\ \text{cis}\ \frac{\pi}{12}$

e $\frac{1}{z_1 z_2} = \frac{1}{4\sqrt{2}}\ \text{cis}\left(-\frac{19\pi}{12}\right) = \frac{\sqrt{2}}{8}\ \text{cis}\ \frac{5\pi}{12}$

$\therefore -\frac{1}{z_1 z_2} = \frac{\sqrt{2}}{8}\ \text{cis}\ \frac{17\pi}{12}$

3 **a** $4 = 4\ \text{cis}\ 0 \quad \sqrt{3} + \mathrm{i} = 2\ \text{cis}\ \frac{\pi}{6}$

$\frac{4}{\sqrt{3} + \mathrm{i}} = 2\ \text{cis}\left(-\frac{\pi}{6}\right) = 2\ \text{cis}\ \frac{11\pi}{6}$

b $2 - 2\mathrm{i} = 2\sqrt{2}\ \text{cis}\ \frac{7\pi}{4} \quad \sqrt{6} + \sqrt{2}\mathrm{i} = 2\sqrt{2}\ \text{cis}\ \frac{\pi}{6}$

$\frac{2 - 2\mathrm{i}}{\sqrt{6} + \sqrt{2}\mathrm{i}} = \text{cis}\ \frac{19\pi}{12}$

c $1 = \text{cis}\ 0\ (\sqrt{21} - \sqrt{7}\mathrm{i})^* = \sqrt{21} + \sqrt{7}\mathrm{i} = 2\sqrt{7}\ \text{cis}\ \frac{\pi}{6}$

$\frac{1}{\left(\sqrt{21} - \sqrt{7}\mathrm{i}\right)^*} = \frac{1}{2\sqrt{7}}\ \text{cis}\left(-\frac{\pi}{6}\right) = \frac{\sqrt{7}}{14}\ \text{cis}\ \frac{11\pi}{6}$

4 $z = 2\sqrt{3} - 2\mathrm{i} \quad w = \frac{1 - \mathrm{i}}{2}$

a $|z| = 4\ \arg(z) = -\frac{\pi}{6} \quad z = 4\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$

$|w| = \frac{\sqrt{2}}{2}\ \arg(w) = -\frac{\pi}{4}$

$w = \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$

b $\frac{z}{w} = \frac{8}{\sqrt{2}}\ \text{cis}\ \frac{\pi}{12} = 4\sqrt{2}\ \text{cis}\frac{\pi}{12}$

c $\frac{z}{w} = \frac{2\sqrt{3} - 2\mathrm{i}}{\frac{1 - \mathrm{i}}{2}} = \frac{4\sqrt{3} - 4\mathrm{i}}{1 - \mathrm{i}} \times \frac{1 + \mathrm{i}}{1 + \mathrm{i}}$

$= \frac{4\sqrt{3} + 4\sqrt{3}\mathrm{i} - 4\mathrm{i} + 4}{2}$

$= (2\sqrt{3} + 2) + (2\sqrt{3} - 2)\mathrm{i}$

d $4\sqrt{2}\ \cos\ \frac{\pi}{12} = 2\sqrt{3} + 2$

$\therefore \cos\ \frac{\pi}{12} = \frac{\sqrt{3} + 1}{2\sqrt{2}} = \frac{\sqrt{6} + \sqrt{2}}{4}$

$4\sqrt{2}\ \sin\ \frac{\pi}{12} = 2\sqrt{3} - 2$

$\therefore \sin\ \frac{\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4}$

$\tan\ \frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}} \times \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} - \sqrt{2}} = \frac{6 - 2\sqrt{12} + 2}{4}$

$\tan\ \frac{\pi}{12} = 2 - \sqrt{3}$

Exercise 12E

1 $z_1 = z\ \text{cis}\ \frac{3\pi}{4} \quad z_2 = \sqrt{3} + \mathrm{i} = 2\ \text{cis}\ \frac{\pi}{6}$

a $z_1^* = 2\ \text{cis}\ \frac{5\pi}{4}\ \left(z_1^*\right)^2 = 4\ \text{cis}\ \frac{5\pi}{2} = 4\ \text{cis}\ \frac{\pi}{2}$

$z_2^3 = 8\ \text{cis}\ \frac{\pi}{2}\ \therefore \left(z_1^*\right)^2 \left(z_2\right)^3 = 32\ \text{cis}\ \pi \text{ or } -32$

b $\frac{z_1}{z_2} = \text{cis}\ \frac{7\pi}{12}\ \therefore \left(\frac{z_1}{z_2}\right)^4 = \text{cis}\ \frac{\pi}{3} = \text{cis}\ \frac{\pi}{3}$

c $\frac{z_1^*}{z_2} = \text{cis}\ \frac{\pi}{12}\ \therefore \left(\frac{z_1^*}{-z_2}\right)^3 = \text{cis}\ \frac{3\pi}{12} = \text{cis}\ \frac{3\pi}{4}$

2 $z_1 = 4\,\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}\ z_2 = \frac{1}{2}\,\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}$

a $z_1 z_2 = 2\,\mathrm{e}^{\frac{\pi}{12}\mathrm{i}}$

b $(z_1)^3 = 64\ \mathrm{e}^{-3\frac{\pi}{4}\mathrm{i}}\ (z_2)^{-2} = 4\,\mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}}$

$(z_1)^3\ (z_2)^{-2} = 256\ \mathrm{e}^{-\frac{17\pi}{12}\mathrm{i}} = 256\,\mathrm{e}^{\frac{7\pi}{12}\mathrm{i}}$

c $\frac{z_1}{z_2} = 8\ \mathrm{e}^{-\frac{7\pi}{12}\mathrm{i}}$

d $z_1^* = 4\,\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}\ \therefore\ \frac{z_1^*}{z_2} = 8\,\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}\ \therefore\ \left(\frac{z_1^*}{z_2}\right)^{-3} = \frac{1}{512}\ \mathrm{e}^{\frac{\pi}{4}\mathrm{i}}$

3 $\left(\frac{\cos\theta - i\sin\theta}{\sin\theta + i\cos\theta}\right)^5 = \left(\frac{\cos(-\theta) + \mathrm{i}\sin(-\theta)}{\cos\left(\frac{\pi}{2} - \theta\right) + \mathrm{i}\sin\left(\frac{\pi}{2} - \theta\right)}\right)^5$

$= [\text{cis}\ (-\theta - (\frac{\pi}{2} - \theta))]^5$

$= \left(\text{cis}\left(-\frac{\pi}{2}\right)\right)^5 = (-\mathrm{i})^5 = -\mathrm{i}$

4 $1 + \mathrm{i}\sqrt{3} = 2\ \text{cis}\ \frac{\pi}{3} \quad 1 - \mathrm{i}\sqrt{3} = 2\ \text{cis}\left(-\frac{\pi}{3}\right)$

$(1 + \mathrm{i}\sqrt{3})^n + (1 - \mathrm{i}\sqrt{3})^n = 2^n\ \text{cis}\ \frac{n\pi}{3} + 2^n\ \text{cis}\left(-\frac{n\pi}{3}\right)$

$= 2^n\ [\cos\ \frac{n\pi}{3} + \mathrm{i}\ \sin\ \frac{n\pi}{3} + \cos\left(-\frac{n\pi}{3}\right) + \mathrm{i}\ \sin\left(-\frac{n\pi}{3}\right)]$

$= 2^n\ (\cos\ \frac{n\pi}{3} + \mathrm{i}\ \sin\ \frac{n\pi}{3} + \cos\ \frac{n\pi}{3} - \mathrm{i}\ \sin\ \frac{n\pi}{3})$

$= 2^n\ (2\ \cos\ \frac{n\pi}{3}) = 2^{n+1}\ \cos\ \frac{n\pi}{3},\ n \in \mathbb{N}$ (QED)

5 **a** $(1 - \mathrm{i})^n = [\sqrt{2}\ \text{cis}\left(-\frac{\pi}{4}\right)]^n = 2^{\frac{n}{2}}\ \text{cis}\left(-\frac{n\pi}{4}\right)$

$= 2^{\frac{n}{2}}\ (\cos\ \frac{n\pi}{4} - \mathrm{i}\ \sin\ \frac{n\pi}{4})$

$\sin\ \frac{n\pi}{4} = 0 \Rightarrow \frac{n\pi}{4} = \pi, 2\pi, 3\pi, \ldots$

$\Rightarrow n = 4, 8, 12, \ldots$

$\cos\ \frac{n\pi}{4} < 0 \quad \therefore n = 4$

b $\cos\ \frac{n\pi}{4} = 0 \quad \frac{n\pi}{4} = \frac{\pi}{2} \quad \therefore n = 2$

6 Let $z = r\ \text{cis}\ \theta \quad r\,(r\ \text{cis}\ \theta)^3 = 16$

$r^4\ (\cos\ 3\theta + \mathrm{i}\sin\theta) = 16$

$$\left.\begin{array}{l} \sin 3\theta = 0 \Rightarrow 3\theta = k\pi \Rightarrow \theta = \frac{k\pi}{3} \quad k \in \mathbb{Z} \\ \cos 3\theta = 1 \Rightarrow 3\theta = 2k\pi \Rightarrow \theta = \frac{2k\pi}{3} \quad k \in \mathbb{Z} \end{array}\right\}$$

$\therefore\ \theta = \frac{2k\pi}{3},\ r^4 = 16 \Rightarrow r = 2$

$z = 2\ \text{cis}\left(\frac{2k\pi}{3}\right)$ (i.e. $z = 2\ \text{cis}\ 0,\ 2\ \text{cis}\ \frac{2\pi}{3},\ 2\ \text{cis}\ \frac{4\pi}{3}$)

$z = 2(1 + 0\mathrm{i}),\ 2(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}),\ 2(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i})$

$z = 2,\ -1 + \sqrt{3}\mathrm{i},\ -1 - \sqrt{3}\mathrm{i}$

Exercise 12F

1 **a** $z_3 = 8\ \text{cis}\ \frac{\pi}{3} \Rightarrow z = 2\ \text{cis}\ (\frac{\pi}{9} + \frac{2k\pi}{3})\ k = 0, 1, 2$

$z = 2\ \text{cis}\ \frac{\pi}{9},\ 2\ \text{cis}\ \frac{7\pi}{9},\ 2\ \text{cis}\ \frac{13\pi}{9}$

b $z^4 = (-4\mathrm{i})^2 = -16 = 16\ \text{cis}\ \pi$

$z = 2\ \text{cis}\left(\frac{\pi + 2k\pi}{4}\right) \quad k = 0, 1, 2, 3$

$z = 2 \text{ cis } \frac{\pi}{4}, 2 \text{ cis } \frac{3\pi}{4}, 2 \text{ cis } \frac{5\pi}{4}, 2 \text{ cis } \frac{7\pi}{4}$

$z = 2\left(\frac{\sqrt{2}}{2} + \text{i}\frac{\sqrt{2}}{2}\right), 2\left(-\frac{\sqrt{2}}{2} + \text{i}\frac{\sqrt{2}}{2}\right), 2\left(-\frac{\sqrt{2}}{2} - \text{i}\frac{\sqrt{2}}{2}\right),$

$2\left(\frac{\sqrt{2}}{2} - \text{i}\frac{\sqrt{2}}{2}\right)$

$z = \sqrt{2} + \text{i}\sqrt{2}, -\sqrt{2} + \text{i}\sqrt{2}, -\sqrt{2} - \text{i}\sqrt{2}, \sqrt{2} - \text{i}\sqrt{2}$

c $z^5 = 32\text{e}^{-\pi\text{i}} \Rightarrow z = 2\text{e}^{\frac{-\pi + 2k\pi}{s}\text{i}}$ $k = 0, 1, 2, 3, 4$

$z = 2\text{e}^{-\frac{\pi}{5}\text{i}}, 2\text{e}^{\frac{\pi}{5}\text{i}}, 2\text{e}^{\frac{3\pi}{5}\text{i}}, 2\text{e}\pi^{\text{i}}, 2^{\frac{7\pi}{5}\text{i}}$

2 **a** $z^2 = 1 - \text{i} = \sqrt{2} \text{ cis}\left(-\frac{\pi}{4}\right)$

$z = 2^{\frac{1}{4}} \text{ cis}\left(-\frac{\pi}{8} + \frac{2k\pi}{2}\right) \quad k = 0, 1$

$z = 2^{\frac{1}{4}} \text{ cis}\left(-\frac{\pi}{8}\right), 2^{\frac{1}{4}} \text{ cis } \frac{7\pi}{8}$

b $z^4 = -\sqrt{3} + \text{i} = 2 \text{ cis } \frac{5\pi}{6}$

$z = 2^{\frac{1}{4}} \text{ cis}\left(\frac{5\pi}{24} + \frac{2k\pi}{4}\right) \quad k = 0, 1, 2, 3$

$z = 2^{\frac{1}{4}} \text{ cis } \frac{5\pi}{24}, 2^{\frac{1}{4}} \text{ cis } \frac{17\pi}{24}, 2^{\frac{1}{4}} \text{ cis } \frac{29\pi}{24}, 2^{\frac{1}{4}} \text{ cis } \frac{41\pi}{24}$

c $z^3 = 27\text{e}^{\frac{\pi}{4}\text{i}} \Rightarrow z = 3\text{e}^{\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)^{\text{i}}}$ $k = 0, 1, 2$

$z = 3\text{e}^{\frac{\pi}{12}\text{i}}, 3\text{e}^{\frac{3\pi}{4}\text{i}}, 3\text{e}^{\frac{17\pi}{12}\text{i}}$

d $z^3 - (\sqrt{2} - \sqrt{2}\text{i})\, z = 0$

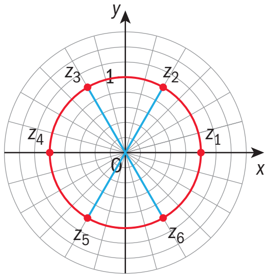
$z = 0 \text{ or } z^2 = \sqrt{2} - \sqrt{2}\text{i} = 2 \text{ cis}\left(-\frac{\pi}{4}\right)$

$z = 0 \text{ or } z = \sqrt{2} \text{ cis}\left(-\frac{\pi}{8} + \frac{2k\pi}{2}\right) \quad k = 0, 1$

$z = 0, \sqrt{2} \text{ cis}\left(-\frac{\pi}{8}\right), \sqrt{2} \text{ cis } \frac{7\pi}{8}$

3 **a** $z^6 = 1 = \text{cis } 0 \Rightarrow z = \text{cis } \frac{2k\pi}{6}, k = 0, 1, 2, 3, 4, 5$

$z = 1, \text{cis } \frac{\pi}{3}, \text{cis } \frac{2\pi}{3}, \text{cis } \pi, \text{cis } \frac{4\pi}{3}, \text{cis } \frac{5\pi}{3}$



b $z_1^3 = 1^3 = 1,\ z_3^3 = (\cos \frac{2\pi}{3})^3 = \text{cis } 2\pi = 1$

$z_5^3 = (\text{cis } \frac{4\pi}{3})^3 = \text{cis } 4\pi = 1$

$\therefore\ z_1^3 = z_3^3 = z_5^3$ (QED)

$z_2^3 = (\text{cis } \frac{\pi}{3})^3 = \text{cis } \pi = -1,\ z_4^3 = (\text{cis } \pi)^3$
$= \text{cis } 3\pi = -1$

$z_6^3 = (\text{cis } \frac{5\pi}{3}) = \text{cis } 5\pi = -1$

$\therefore\ z_2^3 = z_4^3 = z_6^3$ (QED)

z_1, z_3, z_5 are the cube roots of 1 while z_2, z_4, z_6 are the cube roots of -1.

4 $\frac{-1 + \text{i}\sqrt{3}}{4} = \frac{1}{2} \text{ cis } \frac{2\pi}{3}$

the other roots are $\frac{1}{2} \text{ cis}\left(\frac{2\pi}{3} + \frac{2k\pi}{5}\right) k = 1, 2, 3, 4$

ie. $\frac{1}{2} \text{ cis } \frac{16\pi}{15}, \frac{1}{2} \text{ cis } \frac{22\pi}{15}, \frac{1}{2} \text{ cis } \frac{28\pi}{15}, \frac{1}{2} \text{ cis } \frac{4\pi}{15}$

5 $z^4 = -81 = 81 \text{ cis } \pi \Rightarrow z = 3 \text{ cis}\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right)$
$k = 0, 1, 2, 3$

$z = 3 \text{ cis } \frac{\pi}{4}, 3 \text{ cis } \frac{3\pi}{4}, 3 \text{ cis } \frac{5\pi}{4}, 3 \text{ cis } \frac{7\pi}{4}$

$z = \frac{3}{2}\sqrt{2} + \frac{3}{2}\sqrt{2}\text{i}, -\frac{3}{2}\sqrt{2} + \frac{3}{2}\sqrt{2}\text{i}, -\frac{3}{2}\sqrt{2} - \frac{3}{2}\sqrt{2}\text{i},$
$\frac{3}{2}\sqrt{2} - \frac{3}{2}\sqrt{2}\text{i}$

$(z - 3)^4 + 81 = 0 \Rightarrow (z - 3)^4 = -81$

$\therefore z = (\frac{3}{2}\sqrt{2} + 3) + \frac{3}{2}\sqrt{2}\text{i}, (-\frac{3}{2}\sqrt{2} + 3) + \frac{3}{2}\sqrt{2}\text{i},$
$(-\frac{3}{2}\sqrt{2} + 3) - \frac{3}{2}\sqrt{2}\text{i}, (\frac{3}{2}\sqrt{2} + 3) - \frac{3}{2}\sqrt{2}\text{i}$

6 $\text{e}^{0\text{i}\theta} + \frac{1}{3}\text{e}^{2\text{i}\theta} + \frac{1}{9}\text{e}^{4\text{i}\theta}\ \frac{1}{27}\text{e}^{6\text{i}\theta} + \ldots$

$= \frac{1}{1 - \frac{1}{3}\text{e}}\ 2\text{i}\theta = \frac{3}{3 - \text{e}}\ 2\text{i}\theta$

Exercise 12G

1 $z = \text{cis } \alpha$

a $z^n = \cos n\alpha + \text{i}\sin n\alpha \left(\frac{1}{z}\right)^n = (\text{cis } (-\alpha))^n$
$= \text{cis } (-n\alpha)$
$= \cos(-n\alpha) + \text{i} \sin (-n\alpha)$
$= \cos n\alpha - \text{i} \sin n\alpha$

$z^n - \left(\frac{1}{z}\right)^n = \cos n\ \alpha + \text{i} \sin \alpha - (\cos n\alpha - \text{i} \sin n\alpha)$
$= 2 \text{ i} \sin(n\alpha)$ (QED)

b $(z - \frac{1}{z})^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$

$(2\text{i} \sin \alpha)^5 = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$

$32\text{i} \sin^5 \alpha = 2 \text{ i} \sin 5\alpha - 10 \text{ i} \sin 3\alpha +$
$20 \text{ i} \sin \alpha\ (\div 2\text{i})$

$16 \sin^5 \alpha = \sin 5\alpha - 5 \sin 3\alpha + 10 \sin\alpha$

$\int \sin^5 \alpha \, \text{d}\alpha = \frac{1}{16}\int(\sin 5\alpha - 5 \sin 3\alpha + 10 \sin\alpha) \, \text{d}\alpha$

$= \frac{1}{16}(-\frac{1}{5}\cos 5\alpha + \frac{5}{3}\sin 3\alpha - 10 \cos \alpha) + c$

$= -\frac{1}{80}\cos 5\alpha + \frac{5}{48}\sin 3\alpha - \frac{5}{8}\cos \alpha + c$

2 $(\cos \alpha + \text{i} \sin \alpha)^5 = \cos^5 \alpha + 5 \text{ i} \cos^4 \alpha \sin\alpha$
$+ 10\text{i}^2 \cos^3 \alpha \sin^2 \alpha$
$+ 10\text{i}^3 \cos^2 \alpha \sin^3 \alpha$
$+ 5 \text{ i}^4 \cos\alpha \sin^4 \alpha + \text{i}^5 \sin^5 \alpha$

$\therefore \cos^5\alpha + \text{i} \sin^5\alpha = \cos^5 \alpha + 5 \text{ i} \cos^4 \alpha \sin \alpha$
$- 10\cos^3 \alpha \sin^2 \alpha$
$- 10\text{i} \cos^2 \alpha \sin^3 \alpha$
$+ 5 \cos \alpha \sin^4 \alpha + \text{i} \sin^5 \alpha$

Equating real and imaginary parts:

$\cos 5\alpha = \cos^5 \alpha - 10\cos^3\alpha \sin^2 \alpha + 5\cos \alpha \sin^4 \alpha$

$\sin 5\alpha = 5 \cos^4 \alpha \sin\alpha - 10 \cos^2 \alpha \sin^3 \alpha + \sin^5 \alpha$

$\tan 5\alpha = \frac{\sin 5\alpha}{\cos 5\alpha}$

$= \frac{5\cos^4\alpha \sin\alpha - 10\cos^2\alpha \sin^3\alpha + \sin^5\alpha}{\cos^5\alpha - 10\cos^3\alpha \sin^2\alpha + 5\cos\alpha \sin^4\alpha}$

Dividing top and bottom by $\cos^5 \alpha$:

$\tan 5\alpha = \frac{5\tan\alpha - 10\tan^3\alpha + \tan^5\alpha}{1 - 10\tan^2\alpha + 5\tan^4\alpha}$ (QED)

Let $5\alpha = \pi$, $\alpha = \frac{\pi}{5}$

$\tan \pi = 0 \therefore 5 \tan \frac{\pi}{5} - 10 \tan^3 \frac{\pi}{5} + \tan^5 \frac{\pi}{5} = 0$

$\tan \frac{\pi}{5} \neq 0 \therefore 5 - 10 \tan^2 \frac{\pi}{5} + \tan^4 \frac{\pi}{5} = 0$

$\tan^2 \frac{\pi}{5} = \frac{10 \pm \sqrt{100 - 20}}{2}$

$\tan^2 \frac{\pi}{5} = 5 \pm 2\sqrt{5}$

$\tan \frac{\pi}{5} < 1 \therefore \tan^2 \frac{\pi}{5} = 5 - 2\sqrt{5}$

$\therefore \tan \frac{\pi}{5} = \sqrt{5 - 2\sqrt{5}}$ (QED)

3 $(\cos \theta + i \sin \theta)^7 = \cos^7 \theta + 7\cos^6 \theta\, i \sin\theta$
$+ 21 \cos^5 \theta\, i^2 \sin^2 \theta + 35 \cos^4 \theta\, i^3 \sin^3 \theta$
$+ 35 \cos^3 \theta\, i^4 \sin^4 \theta + 21 \cos^2 \theta\, i^5 \sin^5 \theta$
$+ 7 \cos \theta\, i^6 \sin^6 \theta + i^7 \sin^7 \theta \cos 7 \theta + i \sin 7\theta$
$= \cos^7 \theta + 7\, i \cos^6 \theta \sin\theta - 21 \cos^5 \theta \sin^2 \theta$
$- 35\, i \cos^4 \theta \sin^3 \theta + 35 \cos^3 \theta \sin^4 \theta + 21\, i \cos^2$
$\theta \sin^5 \theta - 7 \cos \theta \sin^6 \theta - i \sin^7 \theta$

Equating real parts,

$\cos 7\theta = \cos^7 \theta - 21 \cos^5 \theta \sin^2 \Theta + 35 \cos^3 \theta$
$\sin^4 \theta - 7 \cos \theta \sin^6 \theta$

$\cos 7\theta = \cos^7 \theta - 21 \cos^5 \theta (1 - \cos^2 \theta)$
$+ 35 \cos^3 \theta (1 - \cos^2 \theta)^2 - 7 \cos \theta$
$(1 - \cos^2 \theta)^3$
$= \cos^7 \theta - 21 \cos^5 \theta + 21 \cos^7 \theta + 35 \cos^3 \theta$
$(1 - 2\cos^2 \theta + \cos^4 \theta) - 7 \cos\theta$
$(1 - 3\cos^2 \theta + 3\cos^4 \theta - \cos^6 \theta)$
$= 22 \cos^7 \theta - 21 \cos^5 \theta + 35 \cos^3 \theta - 70$
$\cos^5 \theta + 35 \cos^7 \theta - 7 \cos\theta + 21 \cos^3 \theta$
$- 21 \cos^5 \theta + 7 \cos^7 \theta$

$\therefore \cos 7 \theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta$
$- 7 \cos \theta$ (QED)

$64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta = 1$

$\therefore \cos 7 \theta = 1$

$7\theta = 2k\pi \quad k \in \mathbb{Z}$

$\therefore \theta = \frac{2k\pi}{7} \quad k \in \mathbb{Z}$

4 $z = \cos^2 \theta + \frac{\sin 2\theta}{2} i \; -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

a $|z|^2 = \cos^4 \theta + \frac{\sin^2 2\theta}{2} = \cos^4 \theta + \frac{4\sin^2\theta\cos^2\theta}{4}$
$= \cos^4 \theta + \sin^2 \theta \cos^2 \theta = \cos^2 \theta(\cos^2 \theta$
$+ \sin^2 \theta)$
$= \cos^2 \theta$

$\therefore |z| = \cos \theta$ (QED)

$\arg z = \tan^{-1}\left(\frac{\sin 2\theta}{2\cos^2\theta}\right) = \tan^{-1}\left(\frac{2\sin\theta\cos\theta}{2\cos^2\theta}\right)$

$= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) = \tan^{-1}(\tan \theta) = \theta$ (QED)

b $z = \cos \theta \operatorname{cis} \theta$
$z^2 = \cos^2 \theta \operatorname{cis} 2\theta$

c $|2z^2| = |z| \Rightarrow 2\cos^2 \theta = \cos \theta$
$\cos \theta(2 \cos \theta - 1) = 0$
$\cos \theta = 0$ or $\frac{1}{2}$
$\theta = -\frac{\pi}{2}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}$

5 $w = \frac{z + i}{z + 2}\; z = x + iy$

a $w = \frac{x + i(y + 1)}{(x + 2) + iy} \times \frac{(x + 2) - iy}{(x + 2) - iy}$

$= \frac{x(x + 2) + y(y + 1) + i(y + 1)(x + 2) - ixy}{(x + 2)^2 + y^2}$

$\text{Re}(w) = \frac{x^2 + 2x + y^2 + y}{(x + 2)^2 + y^2}, + \text{I}(w) = \frac{x + 2y + 2}{(x + 2)^2 + y^2}$ (QED)

b **i** $\text{Re}(w) = 1, x^2 + 2x + y^2 + y$
$= x^2 + 4x + 4 + y^2$
$y = 2x + 4\ (\ell_1)$
$\therefore$ the points (x, y) lie on a straight line, gradient $= 2$

ii $\text{Im}(w) = 0, x + 2y + 2 = 0$
$y = -\frac{1}{2} x - 1\ (\ell_2)$
$\therefore$ the points (x, y) lie on a straight line, gradient $= -\frac{1}{2}$
$-\frac{1}{2} \times 2 = -1 \therefore \ell_1$ and ℓ_2 are perpendicular (QED)

c $\text{Arg}(z) = \text{Arg}(w) = \frac{\pi}{4}$
$\therefore \frac{y}{x} = 1$ and $\frac{x + 2y + 2}{x^2 + 2x + y^2 + y} = 1$
$y = x \quad x + 2y + 2 = x^2 + 2x + y^2 + y$
$3x + 2 = 2x^2 + 3x$
$x^2 = 1 \Rightarrow x = \pm 1$
$x = 1, y = 1$ or $x = -1 \therefore |z| = \sqrt{2}$

6 $z = r \operatorname{cis} \theta$

a $p(n)$: $(z^n)^* = (z^*)^n$
$p(1)$: $(z^1)^* = (z^*)^1$
$z^* = z^* \therefore p(1)$ is true
Assume $p(k)$: $(z^k)^* = (z^*)k$
prove $p(k + 1)$: $(z^*)^{k+1} = (z^*)^k z^* = (z^k)^* z^*$
Let $z_1 = r_1 \operatorname{cis} \theta_1\; z_2 = r_2 \operatorname{cis} \theta_2$
$z_1^* = r_1 \operatorname{cis} (-\theta_1)\; z_2^* = r_2 \operatorname{cis} (-\theta_2)$
$z_1 z_2 = r_1 r_2 \operatorname{cis} (\theta_1 + \theta_2)$
$z_1^* z_2^* = r_1 r_2 \operatorname{cis} (-\theta_1 - \theta_2)$
$(z_1 z_2)^* = r_1 r_2 \operatorname{cis}[-(-\theta_1 + -\theta_2)] = r_1 r_2 \operatorname{cis}(-\theta_1 - \theta_2)$
$\therefore z_1^* z_2^* = (z_1 z_2)^*$
$\therefore (z^*)^{k+1} = (z^k z)^* = (z^{k+1})^*$
$\therefore p(k) \Rightarrow p(k + 1)$ and $p(1)$ is true
$\therefore$ by induction $(z^n)^* = (z^*)^n\; n \in \mathbb{Z}^+$ (QED)

b $z = r \text{ cis } \theta,\ z^n = r^n \text{ cis } (n\theta)\ (z^n)^* = r^n \text{ cis } (-n\theta)$

$z^* = r \text{ cis } (-\theta),\ (z^*)^n = r^n \text{ cis } (-n\theta)$

$\therefore\ (z^n)^* = (z^*)^n$ (QED)

Valid for $n \in \mathbb{Z}$ since de Moivre's theorem is valid for $n \in \mathbb{Z}$.

7 **a** $z^3 = 1 = \text{cis } 0$

$z = \text{cis}\left(\frac{2k\pi}{3}\right) \quad k = 0, 1, 2$

$z = \text{cis } 0, \text{ cis } \frac{2\pi}{3}, \text{ cis } \frac{4\pi}{3}$

$z = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}, -\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}$

Let $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}$ and $w^* = -\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}$

$$\frac{1}{1 + w} = \frac{1}{\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}} \times \frac{\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}}{\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}} = \frac{\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}}{1} = -w$$

$\therefore\ \frac{1}{1 + w} = -w$ (QED)

$$\frac{1}{1 + w^*} = \frac{1}{\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}} \times \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}}{\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}} = \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}}{1} = -w^*$$

$\therefore\ \frac{1}{1 + w^*} = -w^*$ (QED)

b $1, -w, -w^*$ are zeros of $p(z)$

$p(z) = (z - 1)(z + w)(z + w^*)$

$= (z - 1)(z^2 + z(w + w^*) + ww^*)$

$w + w^* = -1 \quad ww^* = 1$

$\therefore\ p(z) = (z - 1)(z^2 - z + 1)$

$= z^3 - 2z^2 + 2z - 1$

$\therefore\ a = -2, b = 2, c = -1$

c $p(w) = w^3 - 2w^2 + 2w - 1$

$= 1 - 2w^2 + 2w - 1$

$= -2w^2 + 2w$

$= 2w(1 - w)$

$= (-1 + \sqrt{3}\text{i})\left(\frac{3}{2} - \frac{\sqrt{3}}{2}\text{i}\right)$

$= \frac{1}{2}(-3 + \sqrt{3}\text{i} + 3\sqrt{3}\text{i} + 3)$

$= \frac{1}{2}(4\sqrt{3}\text{i})\ \therefore\ p(w) = 2\sqrt{3}\text{i}$

$p(w^*) = (w^*)^3 - 2(w^*)^2 + 2w^* - 1$

$= 1 - 2(w^*)^2 + 2w^* - 1$

$= 2w^*(1 - w^*)$

$= (-1 - \sqrt{3}\text{i})\left(\frac{3}{2} + \frac{\sqrt{3}}{2}\text{i}\right)$

$= \frac{1}{2}(-3 - \sqrt{3}\text{i} - 3\sqrt{3}\text{i} + 3)$

$= \frac{1}{2}(-4\sqrt{3}\text{i})\ \therefore\ p(w^*)$

$= -2\sqrt{3}\text{i}$

9 $(\sqrt{3} - \text{i})^n = \left(2 \text{ cis}\left(-\frac{\pi}{6}\right)\right)^n = 2^n \text{ cis } \left(-\frac{n\pi}{6}\right)$

$\Rightarrow \cos\left(-\frac{n\pi}{6}\right) > 0$ and $\sin\left(-\frac{n\pi}{6}\right) = 0$

$\frac{n\pi}{6} = k\pi \quad k \in \mathbb{Z}$

$n = 6k \quad k \in \mathbb{Z}$

$\cos\left(\frac{n\pi}{6}\right) > 0 \quad n = 12k \quad k \in \mathbb{Z}$

10 $f(z) = \ln(|z|) + \text{i} \arg(z)$

a $f(\text{i}) = \text{l}n1 + \text{i}\frac{\pi}{2}\ \therefore f(\text{i}) = \text{i}\frac{\pi}{2}$

$f(-\text{i}) = \text{l}n1 - \text{i}\frac{\pi}{2}\ \therefore f(-\text{i}) = -\frac{i\pi}{2}$

$f(1 + \text{i}) = \ln\sqrt{2} + \text{i}\,\frac{\pi}{4} f(1 - \text{i}) = \ln\sqrt{2} - \text{i}\,\frac{\pi}{4}$

b $(f(z))^* = \ln(|z|) - \text{i} \arg(z)$

$|z^*| = |z|$ and $\arg(z^*) = -\arg(z)$

$\therefore f(z^*) = \ln(|z^*|) + \text{i} \arg(z^*)$

$= \ln(|z|) - \text{i} \arg(z)$

$\therefore (f(z))^* = f(z^*)$

c $f(z) = f(z^*)$

$\ln(|z|) + \text{i} \arg(z) = \ln(|z|) - \text{i} \arg(z)$

$2 \text{ i} \arg(z) = 0$

$\therefore \arg(z) = 0$

$\therefore$ z is a real number (QED)

d **i** $\ln(|z|) = 0\ \therefore\ |z| = 1$ and $\arg(z) \neq 0$

$z = x + \text{i}y$ where $x^2 + y^2 = 1$

$(x \neq 1$ since $\arg(z) \neq 0)$

ii $\arg(z) = 0$ and $\ln(|z|) < 0$

$y = 0, x > 0$ and $|z| < 1$

$\therefore z = x\ (0 < x < 1)$

iii $f(z) = 0 \Rightarrow |z| = 1$ and $\arg(z) = 0$

$\therefore\ z = 1$

Review exercise

1 **a** $2\text{cis } \frac{4\pi}{3} = 2(\cos\frac{4\pi}{3} + \text{i} \sin \frac{4\pi}{3}) = -1 - \sqrt{3}\text{i}$

b $\sqrt{2} \text{ cis } 135° = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \text{i}\frac{1}{\sqrt{2}}\right) = -1 + \text{i}$

c $\frac{2}{3\text{cis}\frac{5\pi}{6}} \times \frac{\text{cis}\left(-\frac{5\pi}{6}\right)}{\text{cis}\left(-\frac{5\pi}{6}\right)} = \frac{2}{3} \text{ cis } \left(-\frac{5\pi}{6}\right)$

$= \frac{2}{3}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\text{i}\right) = -\frac{\sqrt{3}}{3} - \frac{1}{3}\text{i}$

2 **a** $5 - 5\text{i} = 5\sqrt{2} \text{ cis } \frac{7\pi}{4}$

b $\frac{2}{1 + \sqrt{3}\text{i}} = \frac{2\text{cis}0}{2\text{cis}\frac{\pi}{3}} = \text{cis}\left(-\frac{\pi}{3}\right)$ or $\text{cis } \frac{5\pi}{3}$

c $\frac{1 - \text{i}}{\sqrt{3} - \text{i}} = \frac{\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)}{2\text{cis}\left(\frac{11\pi}{6}\right)} = \frac{\sqrt{2}}{2} \text{ cis}\left(-\frac{\pi}{2}\right)$ or $\frac{\sqrt{2}}{2} \text{ cis}\left(\frac{11\pi}{12}\right)$

3 $z = 2 \text{ cis } \frac{2\pi}{3}$ $w = 4 \text{ cis } \frac{5\pi}{4}$

a $zw = 8 \text{ cis } \frac{23\pi}{12}, \ \frac{z}{w} = \frac{1}{2} \text{ cis } \left(-\frac{7\pi}{12}\right) \text{ or } \frac{1}{2} \text{ cis } \frac{17\pi}{2}$

$z^2\, w^3 = 4 \text{ cis } \frac{4\pi}{3} \times 64 \text{ cis } \frac{15\pi}{4} = 256 \text{ cis } \frac{13\pi}{12}$

b $z = 2\left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right) = -1 + \mathrm{i}\sqrt{3}$

$w = 4\left(-\frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}\right) = -2\sqrt{2} - 2\sqrt{2}\,\mathrm{i}$

$z + w = (-1 - 2\sqrt{2}) + \mathrm{i}(\sqrt{3} - 2\sqrt{2})$

$z - w = (2\sqrt{2} - 1) + \mathrm{i}(\sqrt{3} + 2\sqrt{2})$

$z^2 = 1 - 2\mathrm{i}\sqrt{3} - 3 = -2 - 2\sqrt{3}\mathrm{i}$

$\therefore \frac{z^2}{w} = \frac{-2 - 2\sqrt{3}\mathrm{i}}{-2\sqrt{2} - 2\sqrt{2}\mathrm{i}} \times \frac{-2\sqrt{2} + 2\sqrt{2}\mathrm{i}}{-2\sqrt{2} + 2\sqrt{2}\mathrm{i}}$

$= \frac{4\sqrt{2} - 4\sqrt{2}\mathrm{i} + 4\sqrt{6}\mathrm{i} + 4\sqrt{6}}{16}$

$\frac{z^2}{w} = \frac{\left(\sqrt{2} + \sqrt{6}\right)}{4} + \frac{\left(\sqrt{6} - \sqrt{2}\right)\mathrm{i}}{4}$

4 $z = a + \mathrm{i}$

a $\arg(z) = \frac{\pi}{3} \therefore \frac{1}{a} = \sqrt{3} \therefore a = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

b $z^2 = a^2 - 1 + 2a\,\mathrm{i}$ $\quad z^2$ is real $\Rightarrow a = 0$

c $|z - 1| = |z - 2\mathrm{i}|$

$|(a - 1) + \mathrm{i}| = |a - \mathrm{i}|$

$(a - 1)^2 + 1 = a^2 + 1 \therefore a^2 - 2a + 1 = a^2$

$\therefore a = \frac{1}{2}$

5 $z^5 = z \Rightarrow z = 0 \text{ or } z^4 = 1$

$z = 0 \text{ or } z^4 = \text{cis } 0$

$z = \text{cis } \frac{2k\pi}{4} \quad k = 0, 1, 2, 3$

$z = 0, \text{cis } 0, \text{cis } \frac{\pi}{2}, \text{cis } \pi, \text{cis } \frac{3\pi}{2}$

$z = 0, 1, \mathrm{i}, -1, -\mathrm{i}$

6 $z = \frac{1}{1 + \mathrm{i}\tan\theta} \times \frac{1 - \mathrm{i}\tan\theta}{1 - \mathrm{i}\tan\theta} = \frac{1 - \mathrm{i}\tan\theta}{\sec^2\theta}$

$z = \cos^2\theta\,\mathrm{i}\sin\theta\cos\theta$

$|z|^2 = \cos^4\theta + \sin^2\theta\cos^2\theta$

$= \cos^2\theta\,(\cos^2\theta + \sin^2\theta)$

$|z|^2 = \cos^2\theta \therefore |z| = \cos\theta$

$\arg(z) = \tan^{-1}\left(-\frac{\sin\theta}{\cos\theta}\right) = \tan^{-1}(\tan(-\theta)) = -\theta$

$\therefore z = \cos\theta \text{ cis } (-\theta)$

7 **a** $\frac{1 - \mathrm{i}}{4} = \frac{\sqrt{2}}{4} \text{ cis } \left(-\frac{\pi}{4}\right)$

$\arg(a + a\mathrm{i}) = \frac{\pi}{4}\ (a > 0) \text{ or } -\frac{3\pi}{4}\ (a < 0)$

since $\frac{1 - \mathrm{i}}{4}$ and $a + a\,\mathrm{i}$ are consecutive nth roots of z and since their arguments differ by $\frac{\pi}{2}$, $n = 4$ and therefore $a = \frac{1}{4}$ or $-\frac{1}{4}$

b The remaining nth roots of z are therefore

$\frac{1 + \mathrm{i}}{4}, \frac{-1 + \mathrm{i}}{4}, \frac{-1 - \mathrm{i}}{4}$

8 $(x + y)(x + wy)(x + w^2 y) = (x + y)(x^2 + w^2 xy + wxy + w^3 y^2)$

$= x^3 + w^2 x^2 y + wx^2 y + w^3 xy^2 + x^2 y + w^2 xy^2 + wxy^2 + w^3 y^3$

$= x^3 + x^2 y(w^2 + w + 1) + xy^2(w^3 + w^2 + w) + w^3 y^3$

$w^3 = 1\ w^3 - 1 = 0$

$(w - 1)(w^2 + w + 1) = 0$

$w \neq 1 \therefore w^2 + w + 1 = 0$

$(x + y)(x + wy)(x + w^2 y) = x^3 + x^2 y(0) + xy^2(0) + y^3$

$= x^3 + y^3$ (QED)

9 $z = -\sqrt{3} - \mathrm{i}$

a $|z| = 2 \ \arg(z) = 210°$

b $z = 2 \sin 210° \ \sqrt[3]{z} = 2^{\frac{1}{3}} \text{ cis } 70° \text{ or } 0.431 + 1.18\,\mathrm{i}$

c $z^n = 2^n \text{ cis } 210\, n°$

z^n is a positive real number

$\therefore \cos 210\, n° > 0$ and $\sin 210\, n° = 0$

$210\, n = 180\, k \quad k \in \mathbb{Z}$

$n = \frac{6}{7}k \quad k \in \mathbb{Z}$

n is an integer $\therefore n = 6, 12, 18, \ldots$

$\cos 210\, n° > 0 \therefore$ smallest positive integer $n = 12$

10 $(\cos\alpha + \mathrm{i}\sin\alpha)^4 = \cos^4\alpha + 4\cos^3\alpha\,\mathrm{i}\sin\alpha - 6\cos^2\alpha\sin^2\alpha - 4\,\mathrm{i}\cos\alpha\sin^3\alpha + \sin^4\alpha$

$\therefore \cos^4\alpha + \mathrm{i}\sin^4\alpha = \cos^4\alpha + 4\mathrm{i}\cos^3\alpha\,\mathrm{i}\sin\alpha - 6\cos^2\alpha\sin^2\alpha - 4\,\mathrm{i}\cos\alpha\sin^3\alpha + \sin^4\alpha$

$\therefore \cos^4\alpha = \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha$

$\sin 4\alpha = 4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha$

$\tan 4\alpha = \frac{\sin 4\alpha}{\cos 4\alpha} = \frac{4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha}{\cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha}$

Dividing top and bottom by $\cos^4\alpha$,

$\tan^4\alpha = \frac{4\tan\alpha - 4\tan^3 x}{1 - 6\tan^2\alpha + \tan^4\alpha}$ (QED)

12 $z + \frac{1}{z} = -1$

a $\left(z + \frac{1}{z}\right)^2 = z^2 + 2 + \frac{1}{z^2} = 1$

$\therefore z^2 + \frac{1}{z^2} = -1$

b $\left(z + \frac{1}{z}\right)^3 = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$

$-1 = z^3 + \frac{1}{z^3} + 3\left(z + \frac{1}{z}\right)$

$\therefore z^3 + \frac{1}{z^3} = 2$

c $\left(z + \frac{1}{z}\right)^5 = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$

$-1 = z^5 + \frac{1}{z^5} + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$

$-1 = z^5 + \frac{1}{z^5} + 10 - 10 \therefore z^5 + \frac{1}{z^5} = -1$

13 a $(x - z)(x - z^*) = x^2 - x(z + z^*) + zz^*$

$z + z^* = 2\text{Re}(z)\ zz^* = |z|^2$

$\therefore (x - z)(x - z^*) = x^2 - 2\text{Re}(z)\, x + |z|^2$ (QED)

b $z^8 = 1 = \text{cis } 0 \Rightarrow z = \frac{\text{cis}2k\pi}{8}$ $k = 0, 1, 2, ...7$

$z = \text{cis } 0, \text{cis } \frac{\pi}{4}, \text{cis } \frac{\pi}{2}, \text{cis } \frac{3\pi}{4}, \text{cis } \pi, \text{cis } \frac{5\pi}{4},$

$\text{cis } \frac{3\pi}{2}, \text{cis } \frac{7\pi}{4}$

$= 1, \frac{\sqrt{2}}{2} + \text{i}\frac{\sqrt{2}}{2}, \text{i}, -\frac{\sqrt{2}}{2} + \text{i}\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2} - \text{i}\frac{\sqrt{2}}{2},$

$-\text{i}, \frac{\sqrt{2}}{2} - \text{i}\frac{\sqrt{2}}{2}$

c $x^8 - 1 = (x - 1)(x + 1)(x - \text{i})(x + \text{i})$

$(x - (\frac{\sqrt{2}}{2} + \text{i}\frac{\sqrt{2}}{2}))(x - (\frac{\sqrt{2}}{2} - \text{i}\frac{\sqrt{2}}{2}))$

$(x - (-\frac{\sqrt{2}}{2} + \text{i}\frac{\sqrt{2}}{2}))(x - (-\frac{\sqrt{2}}{2} - \text{i}\frac{\sqrt{2}}{2}))$

$x^8 - 1 = (x - 1)(x + 1)(x^2 + 1)(x^2 - \sqrt{2}x + 1)$

$(x^2 + \sqrt{2}x + 1)$

14 $w = \text{cis } \alpha\ 0 < \alpha < \frac{\pi}{2}$

a $1 + w = (1 + \cos \alpha) + \text{i} \sin \alpha$

$|1 + w|^2 = (1 + \cos \alpha)^2 + \sin^2 \alpha$

$= 1 + 2 \cos \alpha + \cos^2 \alpha + \sin^2 \alpha$

$= 2 + 2 \cos \alpha$

$= 2 + 2(2\cos^2 \frac{\alpha}{2} - 1)$

$|1 + w|^2 = 4 \cos^2 \frac{\alpha}{2}$

$\therefore\ |1 + w| = 2 \cos \frac{\alpha}{2}$ (QED)

$\arg(1 + w) = \arctan\left(\frac{\sin\alpha}{1 + \cos\alpha}\right)$

$= \arctan\left(\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1 + \left(2\cos^2\frac{\alpha}{2} - 1\right)}\right)$

$= \arctan\left(\tan\frac{\alpha}{2}\right)$

$\therefore \arg(1 + w) = \frac{\alpha}{2}$ (QED)

b $(1 + w)^n = (2 \cos \frac{\alpha}{2} \text{ cis } \frac{\alpha}{2})^n$

$(2 \cos \frac{\alpha}{2})^n \cos \frac{n\alpha}{2} = \text{Re}[(1 + w)^n]$

$= \text{Re}\left[\sum_{k=0}^{n} \binom{n}{k} w^k\right]$

$= \text{Re}[\sum_{k=0}^{n} \binom{n}{k} (\cos k\alpha + \text{i} \sin k\alpha)]$

$= \sum_{k=0}^{n} \binom{n}{k} \cos(k\alpha)$

$\therefore \sum_{k=0}^{n} \binom{n}{k} \cos(k\alpha) \equiv (2 \cos \frac{\alpha}{2})^n \cos (\frac{n\alpha}{2})$ (QED)

Review exercise

1 Using question 11, if z is a zero then z^* is also a zero.

Therefore the 4 roots are $3 \pm \text{i}$ and $\sqrt{2}$ cis $\left(\pm\frac{\pi}{4}\right)$

$\therefore$ the polynomial is

$p(z) = (z - 3 + \text{i})(z - 3 - \text{i})(z - \sqrt{2} \text{ cis } \frac{\pi}{4})$

$\left(z - \sqrt{2} \text{ cis}\left(-\frac{\pi}{4}\right)\right)$

$= [(z - 3)^2 - \text{i}^2]\,[z - 1 - \text{i}]\,[z - 1 + \text{i}]$

$= [(z - 3)^2 - \text{i}^2]\,[(z - 1)^2 - \text{i}^2]$

$= [z^2 - 6z + 9 + 1)(z^2 - 2z + 1 + 1)$

$= (z^2 - 6z + 10)(z^2 - 2z + 2)$

$= z^4 - 8z^3 + 24z^2 - 32z + 20$

Therefore $a = -8$, $b = 24$, $c = -32$, $d = 20$